Solutions of Problems of Final Examination of Math 21b
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1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
11		10
12		10
13		10
Total:		140

• Start by writing your name in the top box and check your section in the box immediately above.

• Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need any additional paper, write your name on it.

• Do not detach pages from this exam packet or un-staple the packet.

• Please write neatly. Answers which are illegible for the grader cannot be given credit.

• No notes, books, calculators, computers, or other electronic aids can be allowed.

• You have three hours to complete your work.

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Problem 1) TF questions (20 points) No justifications needed

1) $\boxed{\text{T}}$ $\boxed{\checkmark\text{F}}$ Let A be a real $n \times n$ matrix and $\vec{a}$ be an element of $\mathbb{R}^n$. Let $\vec{x}(t)$ be the solution of the discrete linear dynamical system $\vec{x}(t+1) = A\vec{x}(t)$ with $\vec{x}(0) = \vec{a}$. Let $\vec{y}(t)$ be the solution of the continuous linear dynamical system $\frac{d\vec{y}(t)}{dt} = A\vec{y}(t)$ with $\vec{y}(0) = \vec{a}$. Then $\vec{x}(t)$ agrees with $\vec{y}(t)$ when t is a positive integer.

$\boxed{\text{Solution}}$ When $n = 1$, the matrix A is just a scalar λ and $y(t) = a\lambda^t$ and $x(t) = ae^{\lambda t}$. The two functions $y(t)$ and $x(t)$ are not the same even when t is a positive integer.

2) $\boxed{\text{T}}$ $\boxed{\checkmark\text{F}}$ Let A be a real $n \times m$ matrix and V be the subspace of $\mathbb{R}^n$ which is spanned by the column vectors of A. Let A^T denote the transpose of A. If the kernel of A is zero, then the matrix of the orthogonal projection from $\mathbb{R}^n$ onto V is $A \left(AA^T\right)^{-1} A^T$.

$\boxed{\text{Solution}}$ AA^T is $n \times n$ and $\left(AA^T\right)^{-1}$ is also $n \times n$ if invertible. When $n \neq m$, the product $A \left(AA^T\right)^{-1}$ makes no sense.

3) $\boxed{\checkmark\text{T}}$ $\boxed{\text{F}}$ Let A be a real $n \times m$ matrix and V be the subspace of $\mathbb{R}^n$ which is spanned by the column vectors of A. Let A^T denote the transpose of A. If the column vectors of A are linearly independent, then the matrix of the orthogonal projection from $\mathbb{R}^n$ onto V is $A \left(A^T A\right)^{-1} A^T$.

4) $\boxed{\text{T}}$ $\boxed{\checkmark\text{F}}$ For any square matrix A and any real number a the determinant $\det(aA)$ of aA is equal to $a(\det A)$.

$\boxed{\text{Solution}}$ When A is $n \times n$, $\det(aA) = a^n \det(A)$.

5) $\boxed{\text{T}}$ $\boxed{\checkmark\text{F}}$ The determinant of any skew-symmetric matrix A is always zero. (Note that by definition A is skew-symmetric if A is equal to the negative $-A^T$ of its transpose A^T.)

$\boxed{\text{Solution}}$ The determinant of the skew-symmetric matrix $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ is 1 and is nonzero.

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6) $\boxed{\checkmark \text{T}}$ $\boxed{\text{F}}$ Let λ be a real number and A and B be real square matrices such that $AB = BA$. If $\vec{x}$ is an eigenvector of A whose eigenvalue is λ, then $B\vec{x}$, if nonzero, is also an eigenvector of A whose eigenvalue is λ.

$\boxed{\text{Solution}}$ $A(B\vec{x}) = B(A\vec{x}) = B(\lambda\vec{x}) = \lambda(B\vec{x})$.

7) $\boxed{\checkmark \text{T}}$ $\boxed{\text{F}}$ If V is a plane in $\mathbb{R}^3$ and A is the 3×3 matrix of the reflection with respect to V, then both $\det(A+I_3) = 0$ and $\det(A-I_3) = 0$. Here I_3 denotes the 3×3 identity matrix.

$\boxed{\text{Solution}}$ Any nonzero vector in the orthogonal complement of V is an eigenvector of A with eigenvalue -1. Hence $\det(A + I_3) = 0$. Any nonzero vector in the plane V is an eigenvector of A with eigenvalue 1. Hence $\det(A - I_3) = 0$.

8) $\boxed{\text{T}}$ $\boxed{\checkmark \text{F}}$ Let $f(x)$ be a continuous function on $[-\pi, \pi]$ of the form

$$\frac{a_0}{\sqrt{2}} + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx)).$$

If $f(x)$ is an even function in the sense that $f(x) = f(-x)$, then $a_n = 0$ for all nonnegative integers n.

$\boxed{\text{Solution}}$ The function which is identically 1 is even, but a_0 is nonzero. Also the function $\cos x$ is even, but a_1 is nonzero.

9) $\boxed{\checkmark \text{T}}$ $\boxed{\text{F}}$ Let $A = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$. The dynamical system $\vec{x}(t + 1) = A\vec{x}(t)$ has the property that $\vec{x}(t) \to 0$ as $t \to \infty$ for all choices of elements $\vec{x}(0)$ of $\mathbb{R}^2$.

$\boxed{\text{Solution}}$ The eigenvalues of A are $\frac{1+i}{2}$ and $\frac{1-i}{2}$. The absolute values of both eigenvalues are < 1.

10) T ☑F   Let $g(x)$ be an infinitely differentiable function on $[-\pi, \pi]$ of the form

$$\frac{a_0}{\sqrt{2}} + \sum_{n=1}^{\infty} \left(a_n \cos(nx) + b_n \sin(nx)\right).$$

Assume that g is odd in the sense that $g(-x) = -g(x)$. Let μ be a positive number. Then

$$f(x,t) = \sum_{n=1}^{\infty} b_n \sin(nx) e^{-n^7 \mu t}$$

is the solution of the heat equation $f_t = \mu f_{xx}$ for $t \geq 0$ and $0 \leq x \leq \pi$ with $f(x,0) = g(x)$ and $f(0,t) = f(\pi,t) = 0$.

Solution   The solution of the heat equation $f_t = \mu f_{xx}$ for $t \geq 0$ and $0 \leq x \leq \pi$ with $f(x,0) = g(x)$ and $f(0,t) = f(\pi,t) = 0$ is

$$f(x,t) = \sum_{n=1}^{\infty} b_n \sin(nx) e^{-n^2 \mu t},$$

where $-n^2\mu t$ should in the exponent instead of $-n^7\mu t$.

11) ☑T F   Let $g(x)$ be an infinitely differentiable function on $[-\pi, \pi]$ of the form $\sum_{n=1}^{\infty} a_n \sin(nx)$. Let $h(x)$ be an infinitely differentiable function on $[-\pi, \pi]$ of the form $\sum_{n=1}^{\infty} b_n \sin(nx)$. Let c be a positive number. Then

$$f(x,t) = \sum_{n=1}^{\infty} \left(a_n \sin(nx) \cos(nct) + \frac{b_n}{nc} \sin(nx) \sin(nct)\right)$$

is the solution of the wave equation $f_{tt} = c^2 f_{xx}$ for $t \geq 0$ and $0 \leq x \leq \pi$ with $f(x,0) = g(x)$ and $f_t(x,0) = h(x)$ and $f(0,t) = f(\pi,t) = 0$.

12) ☑T F   Let A and B be real 2×2 matrices such that the characteristic polynomial $f_A(\lambda) = \det(A - \lambda I_2)$ of A equals to the characteristic polynomial $f_B(\lambda)$ of B. If both roots of $f_A(\lambda) = f_B(\lambda)$ as a polynomial in the variable λ are non-real, then A and B are similar matrices.

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$\boxed{\text{Solution}}$ When the two roots of the common polynomial equation $f_A\left(\lambda\right) = f_B\left(\lambda\right) = 0$ are $a + ib$ and $a - ib$ with a, b real and $b \neq 0$, both A and B are

similar to the rotation-scaling matrix $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$.

13) $\boxed{\text{T}}$ $\boxed{\checkmark\text{F}}$ A matrix A whose eigenvalues all have absolute value 1 must be an orthogonal matrix.

$\boxed{\text{Solution}}$ Both eigenvalues of the upper triangular 2×2 matrix $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ are

1, but its two column vectors are not orthogonal.

14) $\boxed{\checkmark\text{T}}$ $\boxed{\text{F}}$ $\ker(AB) = B^{-1}\ker(A)$ if A is a 4×6 matrix and B is an invertible 6×6 matrix.

$\boxed{\text{Solution}}$ $\vec{x} \in \ker(AB)$ if and only if $AB\vec{x} = \vec{0}$ if and only if $B\vec{x} \in \ker(A)$ if and only if $\vec{x} \in B^{-1}\left(\ker(A)\right)$.

15) $\boxed{\checkmark\text{T}}$ $\boxed{\text{F}}$ A symmetric real matrix A whose eigenvalues are all equal to either 1 or -1 must be an orthogonal matrix.

$\boxed{\text{Solution}}$ Any symmetric real matrix A is orthogonally diagonalizable in the sense that there is an orthogonal matrix S such that $S^{-1}AS = S^T AS$ is a diagonal matrix D. The diagonal entries of D are necessarily the eigenvalues of A and hence is either 1 or -1. Thus $D^{-1} = D$. Since $A = SDS^{-1}$, it follows that $A^{-1} = SD^{-1}S^{-1} = SDS^{-1} = A = A^T$ and A is orthogonal.

16) $\boxed{\checkmark\text{T}}$ $\boxed{\text{F}}$ Let A be a real 9×6 matrix and A^T be its transpose. The orthgonal complement of the image of A^T in $\mathbb{R}^6$ is equal to the kernel of A.

$\boxed{\text{Solution}}$ The orthogonal complement of the image of A^T is equal to the kernel of the transpose of A^T, but the transpose of A^T is just A.

17) $\boxed{\text{T}}$ $\boxed{\checkmark\text{F}}$ Let A be a real 5×7 matrix. If the row vectors of A are linearly independent, then the column vectors of A are also linearly independent.

$\boxed{\text{Solution}}$ There are 7 column vectors, each of which is in $\mathbb{R}^5$. It is not possible to have 7 linearly independent vectors in $\mathbb{R}^5$.

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18) ☑T ☐F   If $\vec{v}$ and $\vec{w}$ are nonzero elements of $\mathbb{R}^n$, then $\vec{v}$ is an eigenvector of the $n \times n$ matrix $\vec{v}\vec{w}^T$, where $\vec{w}^T$ means the transpose of the $n \times 1$ matrix $\vec{w}$.

Solution $\left(\vec{v}\vec{w}^T\right)\vec{v} = \vec{v}\left(\vec{w}^T\vec{v}\right) = \left(\vec{w} \cdot \vec{v}\right)\vec{v}$ so that the nonzero vector $\vec{v}$ is an eigenvector for the $n \times n$ matrix $\vec{v}\vec{w}^T$ with eigenvalue equal to the inner product $\vec{w} \cdot \vec{v}$.

19) ☐T ☑F   Let $\mathcal{C}^\infty\left(\mathbb{R}\right)$ be the linear space of all infinitely differentiable real-valued functions on $\mathbb{R}$. Then the function $f(x) = \sin(5x)$ is an eigenfunction of the linear transformation $T = D^5$ from $\mathcal{C}^\infty\left(\mathbb{R}\right)$ to $\mathcal{C}^\infty\left(\mathbb{R}\right)$, where D sends a function $g(x)$ to its derivative $\frac{dg}{dx}$.

Solution $D^5 f = 5^5\cos(5x)$ which is not a scalar multiple of $f = \sin(5x)$.

20) ☑T ☐F   The matrix $A = \begin{bmatrix} 1 & 0 & 1 & 4 \\ 0 & 3 & 2 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}$ is similar to its transpose

$$A^T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 1 & 2 & -1 & 0 \\ 4 & 1 & 0 & 5 \end{bmatrix}.$$

Solution   An $n \times n$ matrix with n distinct eigenvalues are similar to the diagonal matrix whose diagonal entries are the eigenvalues. Since A is upper triangular, its eigenvalues are its diagonal entries and are all distinct. Its transpose A^T is lower triangular so that the eigenvalues of A^T are its diagonal entries which are the same as the diagonal entries of A. Thus both A and A^T are similar to the same diagonal matrix.

(a) Find a basis of the kernel of the linear transformation from $\mathbb{R}^4$ to $\mathbb{R}^3$ given by the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 4 & 7 \\ 1 & 1 & 2 & 3 \end{bmatrix}.$$

(b) Verify that the rank of A is 2. Find three real numbers c_1, c_2, c_3 not all zero such that

$$c_1 y_1 + c_2 y_2 + c_3 y_3 = 0$$

for all vectors

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

belonging to the image of the matrix A.

$\boxed{\text{Solution}}$ (a) Subtract the first row from the second row in the matrix A to get

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 3 & 6 \\ 1 & 1 & 2 & 3 \end{bmatrix}.$$

Subtract the first row from the second row in the preceding matrix to get

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 3 & 6 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

Divide the second row by 3 in the preceding matrix to get

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

Subtract the second row from the first row in the preceding matrix to get

$$\begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

Subtract the second row from the first row in the preceding matrix to get

$$\begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

An element

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

of $\mathbb{R}^4$ belongs to $\ker(A)$ if and only if

$$\begin{cases} x_1 + x_2 - x_4 = 0, \\ x_3 + 2x_4 = 0, \end{cases}$$

Thus $\vec{x} \in \ker(A)$ if and only if

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -x_2 + x_4 \\ x_2 \\ -2x_4 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix}.$$

The two vectors

$$\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

form a basis of $\ker(A)$.

(b) Let $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$ be the column vectors of A. From the two basis vectors

$$\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

of $\ker(A)$ in Part(a) we have the following two linear relations

$$-\vec{v}_1 + \vec{v}_2 = 0, \quad \vec{v}_1 - 2\vec{v}_3 + \vec{v}_4 = 0.$$

Thus $\vec{v}_1 = \vec{v}_2$ and

$$\vec{v}_3 = \frac{1}{2}\vec{v}_1 + \frac{1}{2}\vec{v}_4 = \frac{1}{2}\vec{v}_2 + \frac{1}{2}\vec{v}_4$$

can be expressed linearly in terms of $\vec{v}_2$ and $\vec{v}_4$. Since

$$\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{v}_4 = \begin{bmatrix} 1 \\ 7 \\ 3 \end{bmatrix}$$

are linearly independent (from the fact that they are not proportional), the dimension of the image of A is 2. The rank of A is 2. We can also see this already from the number of leading 1's in $\mathrm{rref}(A)$ being 2. The coefficients c_1, c_2, c_3 which we seek are to satisfy

$$\begin{cases} c_1 + c_2 + c_3 = 0, \\ c_1 + 7c_2 + 3c_3 = 0. \end{cases}$$

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Consider the coefficient matrix

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 7 & 3 \end{bmatrix}$$

of the above system of two equations with unknowns c_1, c_2, c_3. Subtract the first row from the second row to get

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 6 & 2 \end{bmatrix}.$$

Divide the second row by 6 to get

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & \frac{1}{3} \end{bmatrix}.$$

Subtract the second row from the first row to get

$$\begin{bmatrix} 1 & 0 & \frac{2}{3} \\ 0 & 1 & \frac{1}{3} \end{bmatrix}.$$

The relations to be satisfied by c_1, c_2, c_3 are

$$c_1 + \frac{2}{3} c_3 = 0, \quad c_2 + \frac{1}{3} c_3 = 0.$$

Set $c_3 = 3$ to get $c_1 = -2$ and $c_2 = -1$.

Problem 3) (10 points)

(a) Let

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix} \quad \text{and} \quad \vec{b} = \begin{bmatrix} 2 \\ 2 \\ 1 \\ 4 \end{bmatrix}.$$

Find the least-squares solution $\vec{x}^*$ of the inconsistent system $A\vec{x} = \vec{b}$ of linear equations.

(b) By using (a) or otherwise find the quadratic function $f(x) = a + bx + cx^2$ that best fits the points $(-1, 2), (0, 2), (1, 1), (2, 4)$ in the sense of least squares.

Solution (a) We compute

$$A^T A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 2 \\ 1 & 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 6 \\ 2 & 6 & 8 \\ 6 & 8 & 18 \end{bmatrix}$$

and

$$A^T \vec{b} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 2 \\ 1 & 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 9 \\ 7 \\ 19 \end{bmatrix}$$

We know that the column vectors of A are linearly independent, because according to expansion across the second row the determinant of the submatrix

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

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formed from the first 3 rows of A is equal to -1 times

$$\det \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} = -2.$$

The least-squares solution $\vec{x}^*$ is given by

$$\vec{x}^* = \left(A^T A\right)^{-1} A^T \vec{b}$$

which means that $\vec{x}^*$ is the solution of the system of linear equations

$$\begin{bmatrix} 4 & 2 & 6 \\ 2 & 6 & 8 \\ 6 & 8 & 18 \end{bmatrix} \vec{x}^* = \begin{bmatrix} 9 \\ 7 \\ 19 \end{bmatrix}.$$

Consider the augmented matrix

$$\begin{bmatrix} 4 & 2 & 6 & \vdots & 9 \\ 2 & 6 & 8 & \vdots & 7 \\ 6 & 8 & 18 & \vdots & 19 \end{bmatrix}$$

of the above system of linear equations. Swap the first two rows to get

$$\begin{bmatrix} 2 & 6 & 8 & \vdots & 7 \\ 4 & 2 & 6 & \vdots & 9 \\ 6 & 8 & 18 & \vdots & 19 \end{bmatrix}.$$

Divide the first row by 2 to get

$$\begin{bmatrix} 1 & 3 & 4 & \vdots & \frac{7}{2} \\ 4 & 2 & 6 & \vdots & 9 \\ 6 & 8 & 18 & \vdots & 19 \end{bmatrix}.$$

Subtract 4 times the first row from the second row to get

$$\begin{bmatrix} 1 & 3 & 4 & \vdots & \frac{7}{2} \\ 0 & -10 & -10 & \vdots & -5 \\ 6 & 8 & 18 & \vdots & 19 \end{bmatrix}.$$

Subtract 6 times the first row from the third row to get

$$\begin{bmatrix} 1 & 3 & 4 & \vdots & \frac{7}{2} \\ 0 & -10 & -10 & \vdots & -5 \\ 0 & -10 & -6 & \vdots & -2 \end{bmatrix}.$$

Divide the second row by -10 to get

$$\begin{bmatrix} 1 & 3 & 4 & \vdots & \frac{7}{2} \\ 0 & 1 & 1 & \vdots & \frac{1}{2} \\ 0 & -10 & -6 & \vdots & -2 \end{bmatrix}.$$

Subtract 3 times the second row from the first row to get

$$\begin{bmatrix} 1 & 0 & 1 & \vdots & 2 \\ 0 & 1 & 1 & \vdots & \frac{1}{2} \\ 0 & -10 & -6 & \vdots & -2 \end{bmatrix}.$$

Add 10 times the second row to the third row to get

$$\begin{bmatrix} 1 & 0 & 1 & \vdots & 2 \\ 0 & 1 & 1 & \vdots & \frac{1}{2} \\ 0 & 0 & 4 & \vdots & 3 \end{bmatrix}.$$

Divide the third row by 4 to get

$$\begin{bmatrix} 1 & 0 & 1 & \vdots & 2 \\ 0 & 1 & 1 & \vdots & \frac{1}{2} \\ 0 & 0 & 1 & \vdots & \frac{3}{4} \end{bmatrix}.$$

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Subtract the third row from the second row to get

$$\begin{bmatrix} 1 & 0 & 1 & \vdots & 2 \\ 0 & 1 & 0 & \vdots & \frac{-1}{4} \\ 0 & 0 & 1 & \vdots & \frac{3}{4} \end{bmatrix}.$$

Subtract the third row to the first row to get

$$\begin{bmatrix} 1 & 0 & 0 & \vdots & \frac{5}{4} \\ 0 & 1 & 0 & \vdots & \frac{-1}{4} \\ 0 & 0 & 1 & \vdots & \frac{3}{4} \end{bmatrix}.$$

We obtain

$$\vec{x} = \begin{bmatrix} \frac{5}{4} \\ \frac{-1}{4} \\ \frac{3}{4} \end{bmatrix}.$$

(b) Consider the (possibly inconsistent) system of linear equations

$$\begin{cases} a - b + c = 2, \\ a = 2, \\ a + b + c = 1, \\ a + 2b + 4c = 4, \end{cases}$$

given by

$$\begin{cases} f(-1) = 2, \\ f(0) = 2, \\ f(1) = 1, \\ f(2) = 4, \end{cases}$$

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Thus the least-squares solution $\vec{x}^*$ for the system $A\vec{x} = \vec{b}$ from Part(a) can be used to give the solution

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

for the quadratic function $f(x) = a + bx + cx^2$ that best fits the points $(-1, 2), (0, 2), (1, 1), (2, 4)$ in the sense of least squares. Thus $a = \frac{5}{4}$, $b = \frac{-1}{4}$, and $c = \frac{3}{4}$.

Consider the inner product

$$\langle \vec{v}, \vec{w} \rangle = \vec{v}^T \begin{bmatrix} 1 & 2 \\ 2 & 8 \end{bmatrix} \vec{w}$$

in $\mathbb{R}^2$.

(a) Find all vectors in $\mathbb{R}^2$ perpendicular to $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ with respect to this inner product.

(b) Find an orthonormal basis $\vec{a}, \vec{b}$ of $\mathbb{R}^2$ with respect to this inner product so that the first component of $\vec{a}$ is zero.

$\boxed{\text{Solution}}$ (a) Let

$$\vec{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Then all vectors $\vec{w}$ which is perpendicular to $\vec{v}$ is given by

$$\begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 8 \end{bmatrix} \vec{w} = \vec{0},$$

which is the same as

$$\begin{bmatrix} 2 & 8 \end{bmatrix} \vec{w} = \vec{0}.$$

Thus $\vec{w}$ is equal to a constant times

$$\begin{bmatrix} 4 \\ -1 \end{bmatrix}.$$

(b) The square of the length of

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

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with respect to the given inner product $\langle \vec{v}, \vec{w} \rangle$ is

$$\begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 8 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 8 \end{bmatrix} = 8.$$

The square of the length of

$$\begin{bmatrix} 4 \\ -1 \end{bmatrix}$$

with respect to the given inner product $\langle \vec{v}, \vec{w} \rangle$ is

$$\begin{bmatrix} 4 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 8 \end{bmatrix} \begin{bmatrix} 4 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} = 8.$$

We can set

$$\vec{a} = \frac{1}{\sqrt{8}} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{2\sqrt{2}} \end{bmatrix}$$

and

$$\vec{b} = \frac{1}{\sqrt{8}} \begin{bmatrix} 4 \\ -1 \end{bmatrix} = \begin{bmatrix} \sqrt{2} \\ \frac{-1}{2\sqrt{2}} \end{bmatrix}.$$

Problem 5) (10 points)

(a) Write down the QR decomposition of the matrix

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}$$

(so that $A = QR$, where Q is a matrix whose column vectors are orthonormal and R is an upper triangular matrix with positive diagonal entries).

(b) By using (a) or otherwise find the 4×4 matrix T which represents the orthogonal projection from $\mathbb{R}^4$ onto the subspace of $\mathbb{R}^4$ spanned by the two vectors

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}.$$

$\boxed{\text{Solution}}$ (a) Let $\vec{v}_1$ and $\vec{v}_2$ be the column vectors of the matrix A. Then $\|\vec{v}_1\|^2 = 1 + 1 + 1 + 1 = 4$ and $\|\vec{v}_1\| = 2$. The unit vector $\vec{u}_1$ in the direction of $\vec{v}_1$ is

$$\frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}.$$

The inner product $\vec{v}_2 \cdot \vec{u}_1$ is equal to

$$\frac{1}{2} + 1 + \frac{3}{2} = 3.$$

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The projection $\vec{v}_2^{\perp}$ of $\vec{v}_2$ onto the orthogonal complement of the subspace spanned by $\vec{u}_1$ is given by

$$\vec{v}_2^{\perp} = \vec{v}_2 - (\vec{v}_2 \cdot \vec{u}_1)\,\vec{u}_1 = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix} - 3 \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{-3}{2} \\ \frac{-1}{2} \\ \frac{1}{2} \\ \frac{3}{2} \end{bmatrix}$$

The square of the length of $\vec{v}_2^{\perp}$ is

$$\left(\frac{-3}{2}\right)^2 + \left(\frac{-1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 \left(\frac{3}{2}\right)^2 = \frac{9+1+1+9}{4} = 5.$$

The unit vector $\vec{u}_2$ in the direction of $\vec{v}_2^{\perp}$ is given by

$$\frac{1}{\sqrt{5}}\vec{v}_2^{\perp} = \frac{1}{\sqrt{5}} \begin{bmatrix} \frac{-3}{2} \\ \frac{-1}{2} \\ \frac{1}{2} \\ \frac{3}{2} \end{bmatrix} = \begin{bmatrix} \frac{-3}{2\sqrt{5}} \\ \frac{-1}{2\sqrt{5}} \\ \frac{1}{2\sqrt{5}} \\ \frac{3}{2\sqrt{5}} \end{bmatrix}.$$

We have $\vec{v}_1 = 2\vec{u}_1$ and

$$\vec{v}_2 = (\vec{v}_2 \cdot \vec{u}_1)\,\vec{u}_1 + \vec{v}_2^{\perp} = 3\vec{u}_1 + \vec{v}_2^{\perp} = 3\vec{u}_1 + \sqrt{5}\,\vec{u}_2.$$

In matrix notations,

$$\begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix} = \begin{bmatrix} \vec{u}_1 & \vec{u}_2 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 0 & \sqrt{5} \end{bmatrix}.$$

Thus we have

$$Q = \begin{bmatrix} \vec{u}_1 & \vec{u}_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{-3}{2\sqrt{5}} \\ \frac{1}{2} & \frac{-1}{2\sqrt{5}} \\ \frac{1}{2} & \frac{1}{2\sqrt{5}} \\ \frac{1}{2} & \frac{3}{2\sqrt{5}} \end{bmatrix}$$

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and

$$R = \begin{bmatrix} 2 & 3 \\ 0 & \sqrt{5} \end{bmatrix}.$$

(b) The matrix $T = QQ^T$ is equal to

$$\begin{bmatrix} \frac{1}{2} & \frac{-3}{2\sqrt{5}} \\ \frac{1}{2} & \frac{-1}{2\sqrt{5}} \\ \frac{1}{2} & \frac{1}{2\sqrt{5}} \\ \frac{1}{2} & \frac{3}{2\sqrt{5}} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{-3}{2\sqrt{5}} & \frac{-1}{2\sqrt{5}} & \frac{1}{2\sqrt{5}} & \frac{3}{2\sqrt{5}} \end{bmatrix} = \begin{bmatrix} \frac{7}{10} & \frac{2}{5} & \frac{1}{10} & \frac{-1}{5} \\ \frac{2}{5} & \frac{3}{10} & \frac{1}{5} & \frac{1}{10} \\ \frac{1}{10} & \frac{1}{5} & \frac{3}{10} & \frac{2}{5} \\ \frac{-1}{5} & \frac{1}{10} & \frac{2}{5} & \frac{7}{10} \end{bmatrix}.$$

Let M_n be the $n \times n$ matrix with all 1's along the main diagonal, directly above the main diagonal, and directly below the diagonal, and 0's everywhere else. For example,

$$M_5 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

Let d_n be the determinant of M_n.

(a) Find a formula expressing d_n in terms of d_{n-1} and d_{n-2}.

(b) Find $d_1, d_2, \cdots, d_{10}$. Do you see a pattern?

(c) Find d_{100}.

Solution (a) We expand the determinant across the first row. The minor of the $(1,1)$-th entry in M_n is M_{n-1}. The minor A_{n-1} of the $(1,2)$-th entry in M_n is the same as M_{n-1} except that the $(2,1)$-th entry is replaced by 0. Thus the only nonzero entry on the first column of A_{n-1} is in the $(1,1)$-th position and is equal to 1. The minor of the $(1,1)$-th entry in A_{n-1} is equal to M_{n-2}. The formula expressing d_n in terms of d_{n-1} and d_{n-2} is simply $d_n = d_{n-1} - d_{n-2}$.

(b) Now $d_1 = 1$ and $d_2 = 0$ and

$$\begin{aligned} d_3 &= d_2 - d_1 = -1, \\ d_4 &= d_3 - d_2 = -1, \\ d_5 &= d_4 - d_3 = 0, \\ d_6 &= d_5 - d_4 = 1, \\ d_7 &= d_6 - d_5 = 1, \\ d_8 &= d_7 - d_6 = 0, \\ d_9 &= d_8 - d_7 = -1, \end{aligned}$$

$$d_{10} = d_9 - d_8 = -1.$$

The pattern so far is $1, 0, -1, -1, 0, 1, 1, 0, -1, -1$.

The pattern is clear when we group the sequence into triples. The general description is as follows. We put down first the triple $1, 0, -1$ and then put down the negative of the preceding triple $-1, 0, 1$, and then put down the negative of the preceding triple, and then put down the negative of the preceding triple, etc.

A rigorous mathematical formulation is as follows.

$$d_n = d_{n-1} - d_{n-2} = (d_{n-2} - d_{n-3}) - d_{n-2} = -d_{n-3},$$

which is means that when the index n is increased by 3 the value is the same without a difference in sign. Thus for $j = 0, 1, 2$ we have

$$d_{3k+j} = (-1)^k d_j.$$

Since we start out with $d_1 = 1$, $d_2 = 0$, $d_3 = -1$, we have

$$d_{3k+1} = (-1)^k, \quad d_{3k+2} = 0, \quad d_{3k+3} = -(-1)^k.$$

(c) Since $100 = 3k + 1$ with $k = 99$, we have

$$d_{100} = (-1)^k = (-1)^{99} = -1.$$

Problem 7) (10 points)

Let A be a 2×2 matrix such that its trace is 3 and its determinant is 0. Let λ_1, λ_2 be the eigenvalues of A. Suppose that $\lambda_1 > \lambda_2$.

(a) Find λ_1 and λ_2.

(b) If

$$A \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \lambda_1 \begin{bmatrix} 3 \\ 1 \end{bmatrix} \quad \text{and} \quad A \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \lambda_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix},$$

find the 2×2 matrix A.

(c) Solve the discrete linear dynamical system

$$\vec{y}(t+1) = A\vec{y}(t), \quad \vec{y}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

to obtain a closed formula for $\vec{y}(t)$ for any integer t.

(d) Solve the continuous linear dynamical system

$$\frac{d\vec{x}}{dt} = A\vec{x}, \quad \vec{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

to obtain a closed formula for $\vec{x}(t)$ for any real number t.

(e) Sketch the phase portrait of the continuous linear dynamical system

$$\frac{d\vec{x}}{dt} = A\vec{x}.$$

(f) Determine whether or not the zero state is an asymptotic stable equilibrium of the continuous linear dynamical system

$$\frac{d\vec{x}}{dt} = A\vec{x}.$$

Solution (a) Since the trace $\lambda_1 + \lambda_2$ of A is 3 and the determinant $\lambda_1 \lambda_2$ of A is 0, it follows from $\lambda_1 > \lambda_2$ that $\lambda_1 = 3$ and $\lambda_2 = 0$.

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(b) Let
$$S = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}.$$

Then
$$A = S \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix} S^{-1}.$$

Since
$$S^{-1} = \frac{1}{4} \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix},$$

it follows that
$$\begin{aligned}
A &= \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix} \frac{1}{4} \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix} \\
&= \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{3}{4} & \frac{3}{4} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \frac{9}{4} & \frac{9}{4} \\ \frac{3}{4} & \frac{3}{4} \end{bmatrix}.
\end{aligned}$$

(c) The solution $\vec{y}(t)$ is given by
$$\vec{y}(t) = S \begin{bmatrix} \lambda_1^t & 0 \\ 0 & \lambda_2^t \end{bmatrix} S^{-1} \vec{y}_0.$$

Since
$$S = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad S^{-1} = \frac{1}{4} \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}$$

and $\lambda_1 = 3$, $\lambda_1 = 0$, it follows that
$$\begin{aligned}
\vec{y}(t) &= \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3^t & 0 \\ 0 & 0 \end{bmatrix} \frac{1}{4} \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3^t & 0 \\ 0 & 0 \end{bmatrix} \frac{1}{4} \begin{bmatrix} 1 \\ -1 \end{bmatrix}
\end{aligned}$$

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$$= \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{3^t}{4} \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{3^{t+1}}{4} \\ \frac{3^t}{4} \end{bmatrix}.$$

(d) The solution $\vec{x}(t)$ is given by

$$\vec{x}(t) = S \begin{bmatrix} e^{\lambda_1^t} & 0 \\ 0 & e^{\lambda_2^t} \end{bmatrix} S^{-1} \vec{y}_0.$$

Since

$$S = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad S^{-1} = \frac{1}{4} \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}$$

and $\lambda_1 = 3$, $\lambda_1 = 0$, it follows that

$$\vec{x}(t) = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{3t} & 0 \\ 0 & 1 \end{bmatrix} \frac{1}{4} \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{3t} & 0 \\ 0 & 1 \end{bmatrix} \frac{1}{4} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{e^{3t}}{4} \\ \frac{-1}{4} \end{bmatrix} = \begin{bmatrix} \frac{3e^{3t}+1}{4} \\ \frac{e^{3t}-1}{4} \end{bmatrix}.$$

(e) Let L_1 be the line along the direction of the first eigenvector

$$\begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

and let L_2 be the line along the direction of the second eigenvector

$$\begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

The trajectories are parallel to L_1 and go away from the L_2 as t increases, because the direction fields are along L_1 and point away from L_2.

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(f) Since one of the eigenvalues of A is 3 which is positive, it follows that the zero state is not an asymptotic stable equilibrium of the continuous linear dynamical system

$$\frac{d\vec{x}}{dt} = A\vec{x}.$$

Problem 8) (10 points)

Let P_2 be the vector space (or linear space) consisting of all polynomials with real coefficients and degree ≤ 2.

(a) Show that the set of vectors

$$
\begin{aligned}
f_1(x) &= 1 + x + x^2, \\
f_2(x) &= 2 + 3x + 2x^2, \\
f_3(x) &= 3 + 8x + 2x^2
\end{aligned}
$$

is a basis for P_2.

(b) Express $f(x) = 1 + x + x^2$ as a linear combination of $f_1(x)$, $f_2(x)$, $f_3(x)$.

Solution Let

$$
A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 8 \\ 1 & 2 & 2 \end{bmatrix}.
$$

Part (a) is equivalent to the invertibility of the matrix A. We now determine the invertibility of A and compute at the same time its inverse A^{-1} if it is invertible. Consider

$$
\begin{bmatrix} A \vdots I_3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & \vdots & 1 & 0 & 0 \\ 1 & 3 & 8 & \vdots & 0 & 1 & 0 \\ 1 & 2 & 2 & \vdots & 0 & 0 & 1 \end{bmatrix}.
$$

Subtract the first row from the second row to get

$$
\begin{bmatrix} 1 & 2 & 3 & \vdots & 1 & 0 & 0 \\ 0 & 1 & 5 & \vdots & -1 & 1 & 0 \\ 1 & 2 & 2 & \vdots & 0 & 0 & 1 \end{bmatrix}.
$$

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Subtract the first row from the third row to get

$$\begin{bmatrix} 1 & 2 & 3 & \vdots & 1 & 0 & 0 \\ 0 & 1 & 5 & \vdots & -1 & 1 & 0 \\ 0 & 0 & -1 & \vdots & -1 & 0 & 1 \end{bmatrix}.$$

Subtract 2 times the second row from the first row to get

$$\begin{bmatrix} 1 & 0 & -7 & \vdots & 3 & -2 & 0 \\ 0 & 1 & 5 & \vdots & -1 & 1 & 0 \\ 0 & 0 & -1 & \vdots & -1 & 0 & 1 \end{bmatrix}.$$

Divide the third row by -1 to get

$$\begin{bmatrix} 1 & 0 & -7 & \vdots & 3 & -2 & 0 \\ 0 & 1 & 5 & \vdots & -1 & 1 & 0 \\ 0 & 0 & 1 & \vdots & 1 & 0 & -1 \end{bmatrix}.$$

Add 7 times the third row to the first row to get

$$\begin{bmatrix} 1 & 0 & 0 & \vdots & 10 & -2 & -7 \\ 0 & 1 & 5 & \vdots & -1 & 1 & 0 \\ 0 & 0 & 1 & \vdots & 1 & 0 & -1 \end{bmatrix}.$$

Add -5 times the third row to the second row to get

$$\begin{bmatrix} 1 & 0 & 0 & \vdots & 10 & -2 & -7 \\ 0 & 1 & 0 & \vdots & -6 & 1 & 5 \\ 0 & 0 & 1 & \vdots & 1 & 0 & -1 \end{bmatrix}.$$

The left-half of the above matrix is the reduced row-echelon form of A and its right-half is the inverse of A. Thus we conclude that $\mathrm{rref}(A)$ is the identity

3×3 matrix and

$$A^{-1} = \begin{bmatrix} 10 & -2 & -7 \\ -6 & 1 & 5 \\ 1 & 0 & -1 \end{bmatrix}.$$

For Part (b) if we let $f(x) = c_1 f_1(x) + c_2 f_2(x) + c_3 f_3(x)$, then

$$A \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

and

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = A^{-1} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 & -2 & -7 \\ -6 & 1 & 5 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Hence $c_1 = 1, c_2 = 0, c_3 = 0$ and $f(x) = f_1(x)$. The relation $f(x) = f_1(x)$ one can also see directly without the computation given above.

Problem 9) (10 points)

(a) Find all solutions $f(x)$ to the differential equation

$$f''(x) - 3f'(x) + 2f(x) = 3e^{3x}.$$

(b) Find the unique solution $f(x)$ of the above differential equation which satisfies in addition the initial condition $f(0) = 1$ and $f'(0) = 1$.

Solution (a) First try a solution of the form $e^{\lambda x}$ for the homogeneous equation

$$f''(x) - 3f'(x) + 2f(x) = 0$$

and get the condition

$$\lambda^2 - 2\lambda + 2 = (\lambda - 1)(\lambda - 2) = 0$$

so that λ can be chosen to be 1 or 2. Thus

$$ae^x + be^{2x}$$

for all choices of the constants a and b represent all the solutions of the equation

$$f''(x) - 3f'(x) + 2f(x) = 0.$$

Next try find a particular solution of

$$f''(x) - 3f'(x) + 2f(x) = 3e^{3x}$$

of the form ce^{3x} and get the condition

$$9ce^{3x} - 9ce^{3x} + 2ce^{3x} = 3e^{3x}$$

so that $c = \frac{3}{2}$. Thus

$$ae^x + be^{2x} + \frac{3}{2}e^{3x}$$

for all choices of the constants a and b represent all the solutions of the equation

$$f''(x) - 3f'(x) + 2f(x) = 3e^{3x}.$$

(b) Let
$$f(x) = ae^x + be^{2x} + \frac{3}{2}e^{3x}$$

with undetermined coefficients a and b. We get

(1)
$$f(0) = a + b + \frac{3}{2} = 1$$

and

(2)
$$f'(0) = a + 2b + \frac{9}{2} = 1.$$

To solve for a and b from these two equations (1) and (2), we subtract equation (1) from equation (2) and get $b + 3 = 0$ or $b = -3$. From equation (1), we get
$$a = 1 - \frac{3}{2} - b = \frac{5}{2}.$$

Thus the unique solution $f(x)$ of the equation
$$f''(x) - 3f'(x) + 2f(x) = 3e^{3x}.$$

is given by
$$f(x) = \frac{5}{2}e^x - 3e^{2x} + \frac{3}{2}e^{3x}.$$

Problem 10) (10 points)

Trignometric Identity

$$2\cos(nx)\cos(my) = \cos(nx - my) + \cos(nx + my)$$

(a) Find the Fourier series $\frac{a_0}{\sqrt{2}} + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx))$ for the function $\cos\left(\frac{x}{2}\right)$ on the interval $-\pi \le x \le \pi$.

(b) Use Parseval's identity to find a closed formula for

$$\sum_{n=1}^{\infty} \frac{1}{(4n^2 - 1)^2}.$$

Solution

$$a_n = \frac{2}{\pi} \int_0^\pi \cos(nx) \cos\left(\frac{x}{2}\right) = \frac{1}{\pi} \int_0^\pi \left(\cos\left(\left(n - \frac{1}{2}\right) x\right) + \cos\left(\left(n + \frac{1}{2}\right) x\right) \right) dx$$

$$= \frac{1}{\pi} \left[\frac{1}{n - \frac{1}{2}} \sin\left(\left(n - \frac{1}{2}\right) x\right) + \frac{1}{n + \frac{1}{2}} \sin\left(\left(n + \frac{1}{2}\right) x\right) \right]_{x=0}^\pi.$$

Since

$$\sin\left(\left(n \pm \frac{1}{2}\right) \pi\right) = \pm(-1)^n,$$

it follows that

$$a_n = \frac{1}{\pi} \left(\frac{(-1)^{n+1}}{n - \frac{1}{2}} - \frac{(-1)^{n+1}}{n + \frac{1}{2}} \right) = \frac{(-1)^{n+1}}{\pi} \frac{1}{n^2 - \frac{1}{4}} = \frac{(-1)^{n+1}}{\pi} \frac{4}{4n^2 - 1}$$

for $n \ge 1$.

$$a_0 = \frac{2}{\pi} \int_0^\pi \cos\left(\frac{x}{2}\right) \frac{1}{\sqrt{2}} dx = \frac{\sqrt{2}}{\pi} \left[2 \sin\left(\frac{x}{2}\right) \right]_{x=0}^\pi = \frac{2\sqrt{2}}{\pi}.$$

Parseval's identity reads

$$\frac{2}{\pi} \int_0^\pi \cos^2\left(\frac{x}{2}\right) dx = a_0^2 + \sum_{n=1}^{\infty} a_n^2.$$

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Thus

$$\frac{8}{\pi^2} + \frac{16}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(4n^2 - 1)^2} = \frac{2}{\pi} \int_0^{\pi} \cos^2\left(\frac{x}{2}\right) dx$$

$$= \frac{1}{\pi} \int_0^{\pi} (1 + \cos x)\, dx = \frac{1}{\pi} \left[x + \sin x\right]_{x=0}^{\pi} = 1$$

and

$$\sum_{n=1}^{\infty} \frac{1}{(4n^2 - 1)^2} = \frac{\pi^2}{16}\left(1 - \frac{8}{\pi^2}\right) = \frac{\pi^2 - 8}{16}.$$

(a) Find the eigenvalues and eigenvectors for $\begin{bmatrix} 1 & 28 \\ 7 & -20 \end{bmatrix}$.

(b) Use (a) to find a matrix A such that $A^3 = \begin{bmatrix} 1 & 28 \\ 7 & -20 \end{bmatrix}$.

Solution The eigenvectors of $\begin{bmatrix} 1 & 28 \\ 7 & -20 \end{bmatrix}$ are

$$\begin{bmatrix} 4 \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

and their respectively eigenvalues are 8 and -27 so that

$$\begin{bmatrix} 1 & 28 \\ 7 & -20 \end{bmatrix} = SDS^{-1},$$

where

$$S = \begin{bmatrix} 4 & 1 \\ 1 & -1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 8 & 0 \\ 0 & -27 \end{bmatrix}.$$

Let $E = \begin{bmatrix} 4 & 1 \\ 1 & -1 \end{bmatrix}$ and set $A = SES^{-1}$. Then

$$A^3 = \left(SES^{-1} \right)^3 = SE^3 S^{-1} = SDS^{-1} = \begin{bmatrix} 1 & 28 \\ 7 & -20 \end{bmatrix}.$$

Since the inverse S^{-1} of S is given by

$$S^{-1} = \frac{1}{-5} \begin{bmatrix} -1 & -1 \\ -1 & 4 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 1 & 1 \\ 1 & -4 \end{bmatrix},$$

it follows that one solution for A is

$$A = SES^{-1} = \begin{bmatrix} 4 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix} \frac{1}{5} \begin{bmatrix} 1 & 1 \\ 1 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & -3 \\ 2 & 3 \end{bmatrix} \frac{1}{5} \begin{bmatrix} 1 & 1 \\ 1 & -4 \end{bmatrix}$$

$$= \frac{1}{5} \begin{bmatrix} 5 & 20 \\ 5 & -10 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 1 & -2 \end{bmatrix}.$$

Problem 12) (10 points)

Let $\mathcal{C}^\infty$ be the linear space of real-valued infinitely differentiable functions on $\mathbb{R}$. Consider the linear transformation T from $\mathcal{C}^\infty$ to $\mathcal{C}^\infty$ given by $T(f) = \frac{df}{dx} - f$ for any real-valued infinitely differentiable function $f(x)$ on $\mathbb{R}$.

(a) Show that

$$f(x) = e^x \int_{s=0}^{x} e^{-s}g(s)ds$$

is a solution to the differential equation $\frac{df}{dx} - f = g$ for g in $\mathcal{C}^\infty$.

(b) What is the image and what is the kernel of T?

(c) What is the image and what is the kernel of T^2?

(d) What is the image and what is the kernel of T^6?

Solution (a) To verify that

$$f(x) = e^x \int_{s=0}^{x} e^{-s}g(s)ds$$

satisfies the differential equation $\frac{df}{dx} - f = g$, we differentiate directly and get

$$\frac{df}{dx} - f = \frac{d}{dx}\left(e^x \int_{s=0}^{x} e^{-s}g(s)ds\right) - e^x \int_{s=0}^{x} e^{-s}g(s)ds$$

$$= e^x \int_{s=0}^{x} e^{-s}g(s)ds + e^x e^{-x}g(x) - e^x \int_{s=0}^{x} e^{-s}g(s)ds = g(x).$$

(b) Part(a) shows that the image of T is the set of all infinitely differentiable functions g. The kernel of T consists of all solutions of the homogeneous differential equation

$$f' - f = 0$$

and is the set of all functions

$$ae^x$$

for all choices of the constant a.

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(c) Since the image of T is the set of all infinitely differentiable functions g, the image of T^2 also is the set of all infinitely differentiable functions g. The kernel of T consists of solutions of the homogeneous differential equation

$$f'' - 2f' + f = (D - 1)^2 f = 0$$

and is equal to the set of all functions

$$ae^x + bxe^x$$

for all choices of the constants a and b.

(d) Since the image of T is the set of all infinitely differentiable functions g, the image of T^6 also is the set of all infinitely differentiable functions g. The kernel of T consists of solutions of the homogeneous differential equation

$$(D - 1)^6 f = 0$$

and is equal to the set of all functions

$$\sum_{k=1}^{6} a_k x^{k-1} e^x$$

for all choices of the constants $a_1, \cdots, a_6$.

Problem 13) (10 points)

Trignometric Identities

$$2\cos(nx)\cos(my) = \cos(nx - my) + \cos(nx + my)$$

$$2\sin(nx)\sin(my) = \cos(nx - my) - \cos(nx + my)$$

$$2\sin(nx)\cos(my) = \sin(nx + my) + \sin(nx - my)$$

(a) Let $g(x)$ be the function on $[-\pi, \pi]$ defined by

$$g(x) = \begin{cases} |\sin(2x)| & \text{for } 0 \le x \le \pi \\ \\ -|\sin(2x)| & \text{for } -\pi \le x \le 0. \end{cases}$$

Determine the coefficients a_0, a_n, b_n (for $n \ge 1$) of the Fourier series

$$\frac{a_0}{\sqrt{2}} + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx))$$

of the function $g(x)$ on $[-\pi, \pi]$.

(b) Let $\mu > 0$. Use (a) to find the solution $f(x, t)$ of the heat equation $f_t = \mu f_{xx}$ for $t \ge 0$ and $0 \le x \le \pi$ with the initial condition $f(x, 0) = |\sin(2x)|$ and $f(0, t) = f(\pi, t) = 0$.

(c) Let $c > 0$. Use (a) to solve the solution of the wave equation $f_{tt} = c^2 f_{xx}$ for $t \ge 0$ and $0 \le x \le \pi$ with the initial condition $f(x, 0) = |\sin(2x)|$ and $f_t(x, 0) = 0$ and $f(0, t) = f(\pi, t) = 0$.

(d) Let $c > 0$. Use (a) to solve the solution of the wave equation $f_{tt} = c^2 f_{xx}$ for $t \ge 0$ and $0 \le x \le \pi$ with the initial condition $f(x, 0) = 0$ and $f_t(x, 0) = |\sin(2x)|$ and $f(0, t) = f(\pi, t) = 0$.

Solution (a) Since $g(x)$ as a function on $[-\pi, \pi]$ is odd, all a_0, a_n are zero for $n \ge 1$. To compute b_n for $n \ge 1$, we use the formula

$$b_n = \frac{2}{\pi} \int_0^{\pi} |\sin(2x)| \sin(nx) dx$$

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$$= \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \sin{(2x)} \sin(nx) dx - \frac{2}{\pi} \int_{\frac{\pi}{2}}^{\pi} \sin{(2x)} \sin(nx) dx.$$

We now use the trigonometric identity

$$\sin\theta \sin\varphi = \frac{1}{2} \left(\cos{(\theta - \varphi)} - \cos{(\theta + \varphi)} \right)$$

and write

$$\sin{(2x)} \sin(nx) = \frac{1}{2} \left(\cos{((n-2)x)} - \cos{((n+2)x)} \right)$$

which we integrate to get

$$\int \sin{(2x)} \sin(nx) dx = \int \frac{1}{2} \left(\cos{((n-2)x)} - \cos{((n+2)x)} \right) dx$$
$$= \frac{1}{2(n-2)} \sin{((n-2)x)} - \frac{1}{2(n+2)} \sin{((n+2)x)} + \text{constant}$$

for $n \neq 2$ and

$$\int \sin{(2x)} \sin(2x) dx = \int \frac{1}{2} \left(1 - \cos{(4x)} \right) dx = \frac{x}{2} - \frac{1}{8} \sin{(4x)} + \text{constant}.$$

Thus for $n \neq 2$

$$b_n = \frac{2}{\pi} \left[\frac{1}{2(n-2)} \sin{((n-2)x)} - \frac{1}{2(n+2)} \sin{((n+2)x)} \right]_{x=0}^{x=\frac{\pi}{2}}$$
$$- \frac{2}{\pi} \left[\frac{1}{2(n-2)} \sin{((n-2)x)} - \frac{1}{2(n+2)} \sin{((n+2)x)} \right]_{x=\frac{\pi}{2}}^{x=\pi}$$
$$= \frac{4}{\pi} \left(\frac{1}{2(n-2)} \sin{\left((n-2)\frac{\pi}{2}\right)} - \frac{1}{2(n+2)} \sin{\left((n+2)\frac{\pi}{2}\right)} \right),$$

because

$$\sin{((n-2)x)} = 0 \quad \text{for} \quad x = 0 \quad \text{and} \quad x = \pi.$$

Now

$$\sin{((n\pm 2)x)} \bigg|_{x=\frac{\pi}{2}} = \sin{\left((n\pm 2)\frac{\pi}{2}\right)} = \sin{\left(\frac{n\pi}{2} \pm \pi\right)} = -\sin{\left(\frac{n\pi}{2}\right)}$$

which is equal to $-(-1)^{\frac{n-1}{2}}$ when n is odd and is equal to 0 when n is even. Thus for $n = 2k+1$ we have

$$b_n = \frac{4}{\pi} \left(\frac{1}{2(n-2)} - \frac{1}{2(n+2)} \right) (-1)^{k+1} = \frac{8(-1)^{k+1}}{\pi (n^2 - 4)}$$

and $b_n = 0$ when $n \neq 2$ is even. To computer b_2, we use

$$b_2 = \frac{2}{\pi} \left[\frac{x}{2} - \frac{1}{8} \sin (4x) \right]_{x=0}^{x=\frac{\pi}{2}} - \frac{2}{\pi} \left[\frac{x}{2} - \frac{1}{8} \sin (4x) \right]_{x=\frac{\pi}{2}}^{x=\pi}$$

and get $b_2 = 0$. The Fourier series of $g(x)$ on $[-\pi, \pi]$ is given by

$$g(x) = \sum_{k=1}^{\infty} \frac{8(-1)^{k+1}}{\pi ((2k+1)^2 - 4)} \sin ((2k+1)x).$$

(b) Since the solution heat equation $f_t = \mu f_{xx}$ on $[0, \pi]$ with the initial condition $f(x, 0) = |\sin (2x)|$ and $f(0, t) = f(\pi, t) = 0$. is given by

$$f(x, t) = \sum_{n=1}^{\infty} b_n \sin(nx) e^{-n^2 \mu t}$$

when

$$g(x) = \sum_{n=1}^{\infty} b_n \sin(nx),$$

it follows from

$$g(x) = \sum_{k=1}^{\infty} \frac{8(-1)^{k+1}}{\pi ((2k+1)^2 - 4)} \sin ((2k+1)x)$$

that

$$f(x, t) = \sum_{k=1}^{\infty} \frac{8(-1)^{k+1}}{\pi ((2k+1)^2 - 4)} \sin ((2k+1)x) \, e^{-(2k+1)^2 \mu t}.$$

(c) Since the solution of the wave equation $f_{tt} = c^2 f_{xx}$ for $t \geq 0$ and $0 \leq x \leq \pi$ with the initial condition $f(x, 0) = |\sin (2x)|$ and $f_t (x, 0) = 0$ and $f(0, t) = f(\pi, t) = 0$ is given by

$$f(x, t) = \sum_{n=1}^{\infty} b_n \sin(nx) \cos(nct)$$

when

$$g(x) = \sum_{n=1}^{\infty} b_n \sin(nx),$$

it follows from

$$g(x) = \sum_{k=1}^{\infty} \frac{8(-1)^{k+1}}{\pi \left((2k+1)^2 - 4\right)} \sin\left((2k+1)x\right)$$

that

$$f(x,t) = \sum_{k=1}^{\infty} \frac{8(-1)^{k+1}}{\pi \left((2k+1)^2 - 4\right)} \sin\left((2k+1)x\right) \cos((2k+1)ct).$$

(d) Since the solution of the wave equation $f_{tt} = c^2 f_{xx}$ for $t \geq 0$ and $0 \leq x \leq \pi$ with the initial condition $f(x,0) = 0$ and $f_t(x,0) = |\sin(2x)|$ and $f(0,t) = f(\pi,t) = 0$ is given by

$$f(x,t) = \sum_{n=1}^{\infty} \frac{b_n}{nc} \sin(nx) \cos(nct)$$

when

$$g(x) = \sum_{n=1}^{\infty} b_n \sin(nx),$$

it follows from

$$g(x) = \sum_{k=1}^{\infty} \frac{8(-1)^{k+1}}{\pi \left((2k+1)^2 - 4\right)} \sin\left((2k+1)x\right)$$

that

$$f(x,t) = \sum_{k=1}^{\infty} \frac{8(-1)^{k+1}}{\pi c(2k+1)\left((2k+1)^2 - 4\right)} \sin\left((2k+1)x\right) \sin((2k+1)ct).$$