

**Review Material for the Final Examination of Math 21b**  
**January 17, 2008**

- (1) Coefficient matrix and augmented matrix of a system of linear equations.
- (2) Reduced row-echelon form of a matrix characterized by three conditions:
  - a.* If a row has nonzero entries, then the first nonzero entry is 1, called the leading 1 in this row.
  - b.* If a column contains a leading 1, then all other entries in that column are zero.
  - c.* If a row contains a leading 1, then each row above contains a leading 1 further to the left.
- (3) Reduction of a matrix to reduced row-echelon form by using three kinds of row operations (swapping rows, multiplying a row by a nonzero number, and adding a multiple of a row to another).
- (4) Use reduction to reduced row-echelon form to determine whether a system of linear equations is inconsistent, uniquely solvable, or solvable with an infinite number of solutions and to give a general solution (if it exists) by using free variables.
- (5) Use reduction to reduced row-echelon form to determine whether a square matrix is invertible and to find its inverse if it is invertible.
- (6) Determinant of a  $2 \times 2$  matrix. Formula for the inverse matrix of a  $2 \times 2$  matrix with nonzero determinant.
- (7) The span of a set of vectors. Redundant vectors in a sequence of vectors. Linear dependence and independence of a set of vectors. Subspaces. Bases. Dimension.
- (8) Determine the rank and the nullity of a matrix. Find a basis for the image and for the kernel of a matrix. Rank-Nullity Theorem.
- (9) Equivalence conditions for the invertibility of an  $n \times n$  matrix  $A$ : unique solvability of  $A\vec{x} = \vec{b}$ ,  $\text{rref}(A) = I_n$ ,  $\text{rank}(A) = n$ ,  $\text{im}(A) = \mathbb{R}^n$ ,  $\text{ker}(A) = \{\vec{0}\}$ , column vectors forming a basis of  $\mathbb{R}^n$ , column vectors spanning  $\mathbb{R}^n$ , column vectors linearly independent.

(10) Special linear transformations: rotations, dilations, projections (onto a line or a plane), reflections, and shears (horizontal and vertical).

(11) The column vectors of the matrix of a linear transformation equal to its images of the standard vectors.

(12) Relation between matrix multiplication and the composition of linear transformations.

(13) Coordinates with respect a basis of a subspace. Matrix of a linear transformation with respect to a basis. Relation of matrices of the same linear transformation with respect to two different bases. Similar matrices. Powers of similar matrices. Similarity as an equivalence relation. The vector  $x$  and its new coordinates  $\vec{y}$  with respect to a basis  $\vec{v}_1, \dots, \vec{v}_n$  are related by

$$\vec{x} = \sum_{k=1}^n y_k \vec{v}_k = [\vec{v}_1 \quad \dots \quad \vec{v}_n] \begin{bmatrix} y_1 \\ \cdot \\ \cdot \\ y_n \end{bmatrix} = S\vec{y},$$

where

$$S = [\vec{v}_1 \quad \dots \quad \vec{v}_n], \quad \vec{y} = \begin{bmatrix} y_1 \\ \cdot \\ \cdot \\ y_n \end{bmatrix}.$$

A matrix  $A$  of a linear transformation  $T$  from  $\mathbb{R}^n$  to  $\mathbb{R}^n$  is related to the new matrix

$$B = \begin{bmatrix} b_{11} & \dots & b_{1n} \\ \cdot & \dots & \cdot \\ \cdot & \dots & \cdot \\ \cdot & \dots & \cdot \\ b_{n1} & \dots & b_{nn} \end{bmatrix}.$$

representing  $T$  with respect to  $\vec{v}_1, \dots, \vec{v}_n$  by  $AS = SB$ , because

$$A\vec{v}_k = \sum_{j=1}^n b_{jk} \vec{v}_j = [\vec{v}_1 \quad \dots \quad \vec{v}_n] \begin{bmatrix} b_{1k} \\ \cdot \\ \cdot \\ b_{nk} \end{bmatrix}$$

and

$$A [\vec{v}_1 \quad \cdots \quad \vec{v}_n] = \sum_{k=1}^n b_{jk} \vec{v}_j = [\vec{v}_1 \quad \cdots \quad \vec{v}_n] \begin{bmatrix} b_{11} & \cdots & b_{1n} \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ b_{n1} & \cdots & b_{nn} \end{bmatrix}.$$

(14) Concept of a linear space (also known as a vector space). Addition and scalar multiplication in a linear space and the laws (associativity, commutativity, distributivity, etc.) satisfied by them. Examples of linear spaces: solutions of differential equations, spaces of polynomials, spaces of matrices, etc. Dimension of linear space. Finite and infinite dimension.

(15) Gram-Schmidt process of inductively constructing orthonormal vectors  $u_1, \dots, u_m$  from linearly independent vectors  $\vec{v}_1, \dots, \vec{v}_m$  in  $\mathbb{R}^n$ .  $\vec{v}_j^\perp$  is the orthogonal projection onto the subspace spanned by  $\vec{v}_1, \dots, \vec{v}_{j-1}$  (which is the same as the subspace spanned by  $\vec{u}_1, \dots, \vec{u}_{j-1}$ ) and  $\vec{u}_j$  is the unit vector in the direction of  $\vec{v}_j^\perp$ .

(16) QR decomposition of an  $n \times m$  matrix  $A$  in the form  $QR$ , where  $Q$  is an  $n \times m$  matrix whose column vectors are orthonormal and  $R$  is an  $m \times m$  matrix which is upper triangular with positive diagonal entries.

$$[\vec{v}_1 \quad \cdots \quad \vec{v}_m] = [\vec{u}_1 \quad \cdots \quad \vec{u}_m] R,$$

where  $R$  is the upper triangular  $m \times m$  matrix whose  $(i, j)$ -th entry is  $r_{i,j} = \vec{u}_i \cdot \vec{v}_j$ .

(17) Orthogonal transformations as length-preserving and orthogonality-preserving transformations. Orthonormal set of vectors and orthonormal basis. Pythagorean theorem. Cauchy-Schwarz inequality. Angle between vectors.

(18) Transpose of a matrix. Product of transposes of matrices and inverse of the transpose of a matrix. Symmetric and skew-symmetric matrices. Inner product of two vectors as the matrix product of the transpose of a vector and the other vector:  $\vec{v} \cdot \vec{w} = \vec{v}^T \vec{w}$ . The kernel of the transpose of a matrix as orthogonal complement of the image of the matrix:  $(\text{im}(A))^\perp = \ker(A^T)$ .

(19) Formula  $QQ^T$  for the orthogonal projection onto a subspace spanned by orthonormal vectors which are the column vectors of  $Q$ . Formula  $A(A^T A)^{-1}A^T$  for the orthogonal projection onto a subspace spanned by linearly independent vectors which are the column vectors of  $A$ . Note that the formula  $A(A^T A)^{-1}A^T$  is reduced to  $QQ^T$  when  $A = Q$  and is reduced to the identity matrix when  $A$  is invertible.

(20) Formula  $\vec{x}^* = (A^T A)^{-1}A^T \vec{b}$  for the unique least-squares solution  $\vec{x}^*$  of a (possibly inconsistent) system of linear equations  $A\vec{x} = \vec{b}$  when the condition  $\ker(A) = \{\vec{0}\}$  is satisfied (which is equivalent to the invertibility of  $A^T A$  and also equivalent to the linearly independence of the column vectors of  $A$ ). Application of least-squares solution to finding a polynomial of given degree which best fits a given collection of data by regarding the coefficients of the polynomial as the components of the least-squares solution to the system of equations defined by data fitting.

(21) Determinant defined by induction and expansion down the first column. Determinant computed by expansion down any column and across any row. Formula for the determinant of a  $2 \times 2$  matrix. Formula for the inverse of an invertible  $2 \times 2$  matrix. Formula for the determinant of a  $3 \times 3$  matrix. Formula for the determinant of an upper or lower triangular matrix. Formula for the determinant of an upper or lower triangular partitioned matrix with square matrices as diagonal entries.

(22) Effect of Gauss-Jordan row operations on a determinant. Computation of determinant by Gaussian elimination. Determinant of the transpose of a matrix. Determinant of a product of matrices. Determinant of the inverse of a matrix. Determinants of similar matrices.

(23) Characteristic equation of a square matrix. Trace of a matrix. Eigenvalues, eigenvectors, eigenspaces and their computations. Eigenbasis and diagonalization. Algebraic and geometric multiplicities and their relation. Diagonalization of a matrix whose eigenvalues are all real and distinct. Eigenbasis and diagonalization for a linear transformation of a linear space (with domain and codomain of the linear transformation both equal to the same linear space). Diagonalization of a real symmetric matrix with respect to an orthonormal basis.

(24) Complex eigenvalues. Real and imaginary parts, the complex conjugate, and the absolute value of a complex number. De Moivre's formula

$$(\cos \theta + i \sin \theta)^n = \cos(n\theta) + i \sin(n\theta).$$

Any real  $2 \times 2$  matrix  $A$  with non-real eigenvalues  $a+ib$  is similar to a rotation-scaling matrix. Formula for such a similarity given as follows. Taking the imaginary and real parts of

$$A(\vec{v} + i\vec{w}) = (a + ib)(\vec{v} + i\vec{w})$$

yields

$$\begin{aligned} A\vec{w} &= a\vec{w} + b\vec{v}, \\ A\vec{v} &= -b\vec{w} + a\vec{v} \end{aligned}$$

which can be rewritten in matrix form as

$$A \begin{bmatrix} \vec{w} & \vec{v} \end{bmatrix} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} \vec{w} & \vec{v} \end{bmatrix}.$$

With respect to the new basis  $\vec{w}, \vec{v}$  the real  $2 \times 2$  matrix  $A$  becomes the rotation-scaling matrix

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} = r \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}.$$

(25) Discrete linear dynamical systems. Closed formula for the state vector of the system by using diagonalization with respect to an eigenbasis. For  $A$  with eigenbasis  $\vec{v}_1, \dots, \vec{v}_n$  and eigenvalues  $\lambda_1, \dots, \lambda_n$ , the solution of the discrete linear dynamical system

$$\vec{x}(t+1) = A\vec{x}(t), \quad \vec{x}(0) = \vec{x}_0$$

is

$$\vec{x}(t) = S \begin{bmatrix} \lambda_1^t & 0 & 0 & \dots & 0 \\ 0 & \lambda_2^t & 0 & \dots & 0 \\ \cdot & \cdot & \cdot & \dots & \cdot \\ 0 & 0 & \dots & 0 & \lambda_n^t \end{bmatrix} S^{-1} \vec{x}_0$$

where

$$S = [\vec{v}_1 \quad \dots \quad \vec{v}_n].$$

(26) Discrete trajectories and phase portraits. A phase portrait shows trajectories for various initial states, capturing all the qualitatively different scenarios (pp.301-302).

By definition the zero state  $\vec{0}$  is an asymptotically stable equilibrium for the system

$$\vec{x}(t+1) = A\vec{x}(t), \quad \vec{x}(0) = \vec{x}_0$$

if

$$\lim_{t \rightarrow \infty} \vec{x}(t) = \vec{0}$$

for all its trajectories, that is, for all initial states  $\vec{x}_0$ . This is the case if and only if the absolute values of all the complex eigenvalues of  $A$  are  $< 1$ .

(27) Continuous linear dynamical systems. For  $A$  with eigenbasis  $\vec{v}_1, \dots, \vec{v}_n$  and eigenvalues  $\lambda_1, \dots, \lambda_n$ , the solution of the continuous linear dynamical system

$$\frac{d\vec{x}}{dt} = A\vec{x}, \quad \vec{x}(0) = \vec{x}_0$$

is

$$\vec{x}(t) = S \begin{bmatrix} e^{\lambda_1 t} & 0 & 0 & \dots & 0 \\ 0 & e^{\lambda_2 t} & 0 & \dots & 0 \\ \cdot & \cdot & \cdot & \dots & \cdot \\ 0 & 0 & \dots & 0 & e^{\lambda_n t} \end{bmatrix} S^{-1} \vec{x}_0,$$

where

$$S = [\vec{v}_1 \quad \dots \quad \vec{v}_n].$$

A phase portrait is obtained by looking at the direction field  $A\vec{x}$  and the trajectories (also known as flow lines). See p.397.

(28) Euler's formula

$$e^{i\theta} = \cos \theta + i \sin \theta.$$

For real  $2 \times 2$  matrix  $A$  with eigenvalue  $r(\cos \theta + i \sin \theta)$  and eigenvector  $\vec{v} + i\vec{w}$  with  $r \sin \theta \neq 0$ , the solution of

$$\vec{x}(t+1) = A\vec{x}(t), \quad \vec{x}(0) = \vec{x}_0$$

is

$$\vec{x}(t) = r^t S \begin{bmatrix} \cos(\theta t) & -\sin(\theta t) \\ \sin(\theta t) & \cos(\theta t) \end{bmatrix} S^{-1} \vec{x}_0,$$

where

$$S = [\vec{w} \quad \vec{v}].$$

For the phase portrait, when  $r = 1$  the points  $\vec{x}(t)$  lies on an ellipse. If  $r > 1$  the trajectory spirals outward. If  $r < 1$  the trajectory spirals inward, approaching the origin (p.359).

(29) For real  $2 \times 2$  matrix  $A$  with eigenvalue  $p + iq$  and eigenvector  $\vec{v} + i\vec{w}$  with  $q \neq 0$ , the solution of

$$\frac{d\vec{x}}{dt} = A\vec{x}, \quad \vec{x}(0) = \vec{x}_0$$

is

$$\vec{x}(t) = e^{pt} S \begin{bmatrix} \cos(qt) & -\sin(qt) \\ \sin(qt) & \cos(qt) \end{bmatrix} S^{-1} \vec{x}_0,$$

where

$$S = [\vec{w} \quad \vec{v}].$$

For the phase portrait, when  $p = 0$  the trajectory is an ellipse. If  $p > 0$  the trajectory spirals outward. If  $p < 0$  the trajectory spirals inward, approaching the origin (p.412).

(30) Comparison of the discrete and continuous models. The equation for the discrete model

$$\vec{x}(t+1) = B\vec{x}(t)$$

gives rise to the difference equation

$$\vec{x}(t+1) - \vec{x}(t) = (B - I_n) \vec{x}(t)$$

which corresponds to the following equation for the continuous model

$$\frac{d\vec{x}}{dt} = A\vec{x}.$$

For this consideration  $B$  corresponds to  $I_n + A$  instead of  $A$ .

(31) Homogeneous ordinary linear differential equations with constant coefficients. Use an eigenfunction  $e^{\lambda x}$  (for the operator  $f \mapsto \frac{df}{dx}$ ) with undetermined constant  $\lambda$  to get a solution of the equation

$$a_n f^{(n)} + a_{n-1} f^{(n-1)} + \cdots + a_1 f' + a_0 f = 0$$

by solving for  $\lambda$  in the characteristic equation

$$(*) \quad a_n \lambda^n + a_{n-1} \lambda^{n-1} + \cdots + a_1 \lambda + a_0 = 0.$$

The general solution is a linear combination of

$$e^{\lambda_j x}, x e^{\lambda_j x}, \dots, x^{m_j-1} e^{\lambda_j x} \quad (\text{for } 1 \leq j \leq k),$$

where  $\lambda_1, \dots, \lambda_k$  are all the distinct roots of  $(*)$  with multiplicities respectively  $m_1, \dots, m_k$ .

When  $\lambda = p + iq$  with  $p, q$  real and  $q \neq 0$ , a linear combination of  $e^{\lambda x}$  and  $e^{\bar{\lambda}x}$  is equivalent to a linear combination of  $e^{px} \cos(qx)$  and  $e^{px} \sin(qx)$  due to Euler's formula.

(32) Inhomogeneous ordinary linear differential equations with constant coefficients. To solve the inhomogeneous equation

$$a_n f^{(n)} + a_{n-1} f^{(n-1)} + \cdots + a_1 f' + a_0 f = g,$$

if  $g$  is of the form  $Ax^m e^{\lambda x}$  or  $x^m (Ae^{px} \cos(qx) + Be^{px} \sin(qx))$  (with constants  $A, B$ ) try a particular solution of the same form with different to-be-determined coefficients  $A, B$  (and a different  $m$  when  $\lambda$  or  $p + qi$  is a root of the polynomial equation  $(*)$ ).

When  $g$  is a general function, the general solution of the first-order equation

$$f'(x) - af(x) = g(x)$$

with  $a$  being constant is

$$f(x) = e^{ax} \int e^{-ax} g(x) dx$$

involving an indefinite integral. The solution of the equation of order  $n$

$$(D - \lambda_1)(D - \lambda_2) \cdots (D - \lambda_n) f = g$$

(where  $D = \frac{d}{dx}$  and  $\lambda_1, \dots, \lambda_n$  are constants) can be inductively obtained by solving for  $F$  in the first-order equation

$$(D - \lambda_1) F = g$$

and then solving for  $f$  in the equation of order  $n - 1$

$$(D - \lambda_2) \cdots (D - \lambda_n) f = F.$$

(33) Fourier series and Parseval's identity.

$$\frac{1}{\sqrt{2}}, \cos(nx), \sin(nx)$$

on  $[-\pi, \pi]$  form an orthonormal family with respect to the inner product

$$\langle f, g \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)g(x)dx.$$

Let  $f(x)$  be on  $[-\pi, \pi]$ . Its Fourier coefficients are given by

$$\begin{aligned} a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \frac{1}{\sqrt{2}} dx, \\ a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx, \\ b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) dx. \end{aligned}$$

The Fourier series of  $f$  is

$$\frac{a_0}{\sqrt{2}} + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx)).$$

Parseval's identity is

$$\|f\|^2 = \frac{1}{\pi} \int_{-\pi}^{\pi} (f(x))^2 dx = a_0^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2).$$

Note that we are using here the notations  $a_0, a_n, b_n$  (for  $n \geq 1$ ) in the notes and not the notations  $a_0, b_n, c_n$  (for  $n \geq 1$ ) in Otto Bretscher's book.

(34) Heat Equation. Let  $g(x)$  be a function on  $[0, \pi]$  of the form

$$\sum_{n=1}^{\infty} b_n \sin(nx).$$

Let  $\mu$  be a positive number. Then

$$f(x, t) = \sum_{n=1}^{\infty} b_n \sin(nx) e^{-n^2 \mu t}$$

is the solution of the heat equation  $f_t = \mu f_{xx}$  for  $t \geq 0$  and  $0 \leq x \leq \pi$  with  $f(x, 0) = g(x)$  and  $f(0, t) = f(\pi, t) = 0$ .

Note that the interval for the variable  $x$  is chosen to be  $[0, \pi]$  so that we can extend the function  $g$  on it to an odd function on  $[-\pi, \pi]$  in order to have fewer Fourier coefficients to compute.

(35) Wave Equation. Let  $g(x)$  be a function on  $[0, \pi]$  of the form  $\sum_{n=1}^{\infty} a_n \sin(nx)$ . Let  $h(x)$  be a function on  $[0, \pi]$  of the form  $\sum_{n=1}^{\infty} b_n \sin(nx)$ . Let  $c$  be a positive number. Then

$$f(x, t) = \sum_{n=1}^{\infty} \left( a_n \sin(nx) \cos(nct) + \frac{b_n}{nc} \sin(nx) \sin(nct) \right)$$

is the solution of the wave equation  $f_{tt} = c^2 f_{xx}$  for  $t \geq 0$  and  $0 \leq x \leq \pi$  with  $f(x, 0) = g(x)$  and  $f_t(x, 0) = h(x)$  and  $f(0, t) = f(\pi, t) = 0$ .

Note that the interval for the variable  $x$  is chosen to be  $[0, \pi]$  so that we can extend the two functions  $g$  and  $h$  on it to odd functions on  $[-\pi, \pi]$  in order to have fewer Fourier coefficients to compute.