

Solution of First Midterm of Math 21b, October 24, 2007

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- Start by writing your name in the above box and check your section in the box to the left.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or un-staple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes to complete your work.

1		20
2		10
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9		10
Total:		100

Problem 1) TF questions (20 points) No justifications needed

- 1) T F The plane $x - y + z = 5$ is the kernel of some linear transformation $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$.

Solution The plane $x - y + z = 5$ is not a linear subspace of \mathbb{R}^3 , because, for $\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ and $\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$ both satisfying $x_j - y_j + z_j = 5$ for $j = 1, 2$, their sum $\begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \\ z_1 + z_2 \end{bmatrix}$ does not satisfy $(x_1 + x_2) - (y_1 + y_2) + (z_1 + z_2) = 5$. Thus the plane $x - y + z = 5$ cannot be the kernel of some linear transformation $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$.

- 2) T F For a basis \vec{v}_1, \vec{v}_2 in \mathbb{R}^2 with \vec{v}_1 perpendicular to \vec{v}_2 , the linear transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ whose matrix with respect to the basis \vec{v}_1, \vec{v}_2 is $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is the orthogonal projection onto the line spanned by \vec{v}_1 .

Solution The linear transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ whose matrix with respect to the basis \vec{v}_1, \vec{v}_2 is $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ sends \vec{v}_1 to \vec{v}_1 and sends \vec{v}_2 to $\vec{0}$. Thus it represents the orthogonal projection of \mathbb{R}^2 onto the line spanned by \vec{v}_1 , because \vec{v}_1 is perpendicular to \vec{v}_2 .

- 3) T F A and $\text{rref}(A)$ always have the same rank.

- 4) T F A and $\text{rref}(A)$ always have the same image.

Solution For example, $A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ and $\text{rref}(A) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ do not have the same image, because the image of A is spanned by the vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and the image of $\text{rref}(A)$ is spanned by the vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

- 5) T F A and $\text{rref}(A)$ always have the same kernel.
- 6) T F If A and B are both invertible $n \times n$ matrices, then $A + B$ is invertible.

Solution For example, $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ would give a counter-example, because $A + B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ is not invertible, while both A and B are invertible.

- 7) T F If A and B are both invertible $n \times n$ matrices, then AB is invertible.
- 8) T F If A is an invertible $n \times n$ matrix and B is a $p \times n$ matrix, then BA has the same image as B .
- 9) T F If A is an invertible $n \times n$ matrix and B is a $p \times n$ matrix, then BA has the same kernel as B .

Solution The kernel of BA is equal to $A^{-1}\ker(B)$ which in general is not equal to $\ker(B)$. For example, if $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, then the kernel of B is spanned by the vector $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and the kernel of $BA = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is spanned by the vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

- 10) T F If A is an invertible $n \times n$ matrix and B is a $p \times n$ matrix, then BA has the same rank as B .
- 11) T F If A is an invertible $n \times n$ matrix and B is a $p \times n$ matrix, then BA has the same nullity as B .

- 12) T F There is an invertible matrix S with $S^{-1} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} S = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$.

Solution The condition can be rewritten as $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} S = S \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$. Let $S = \begin{bmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{bmatrix}$. Then $\begin{bmatrix} s_{21} & s_{22} & s_{23} \\ 0 & 0 & 0 \\ s_{31} & s_{32} & s_{33} \end{bmatrix} = \begin{bmatrix} s_{12} & s_{13} & 0 \\ s_{22} & s_{23} & 0 \\ s_{32} & s_{33} & 0 \end{bmatrix}$. From the bottom rows on both sides we conclude that $s_{31} = s_{32} = s_{33} = 0$ and S cannot be invertible, because its last row is zero.

13) T F The nullity of a 4×6 matrix may be 5.

Solution For example, the matrix can be the 4×6 matrix whose entries are all zero except that the entry on the first row and the first column is 1.

14) T F The nullity of a 6×4 matrix may be 5.

Solution The nullity cannot exceed the number of columns.

15) T F There exists a linear transformation T from \mathbb{R}^4 to \mathbb{R}^4 for which $\ker(T) = \text{im}(T)$.

Solution For example, T sends $c_1\vec{e}_1 + c_2\vec{e}_2 + c_3\vec{e}_3 + c_4\vec{e}_4$ to $c_1\vec{e}_3 + c_2\vec{e}_4$ so that both its kernel and its image have \vec{e}_3, \vec{e}_4 as a basis, where $\vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_4$ are the standard vectors of \mathbb{R}^4 .

16) T F There exists a linear transformation T from \mathbb{R}^5 to \mathbb{R}^5 for which $\ker(T) = \text{im}(T)$.

Solution The sum of the nullity and the rank of a matrix must be equal to the number of columns in it. Since the number of columns is 5 which is odd, the dimension of $\ker(T)$ cannot be equal to the dimension of $\text{im}(T)$.

17) T F If A, B are similar $n \times n$ matrices, then $A^4 - A^2 + A$ must be similar to $B^4 - B^2 + B$.

Solution If $A = SBS^{-1}$ for some invertible $n \times n$ matrix S , then we have $A^4 - A^2 + A = S(B^4 - B^2 + B)S^{-1}$, because $A^k = SB^kS^{-1}$ for any positive integer k .

18) T F If the rows of an $n \times n$ matrix form a basis of \mathbb{R}^n , then its columns also form a basis of \mathbb{R}^n .

Solution If the rows of an $n \times n$ matrix A form a basis of \mathbb{R}^n , then the reduced row-echelon form $\text{rref}(A)$ of A must be the $n \times n$ identity matrix I_n , because the three kinds of row operations used to reduce A to $\text{rref}(A)$ do not change the linear independence property of the n row vectors and thus we cannot have any zero row in $\text{rref}(A)$. It follows from $\text{rref}(A) = I_n$ that the kernel of A consists only of the zero vector and the n column vectors of A must be linearly independent.

19) T F There is a 9×9 invertible matrix whose inverse has nullity 3.

Solution The nullity of an invertible matrix must be zero.

20) T F If A, B are $n \times n$ matrices and B is invertible, then there exists a unique matrix X such that $A + BX = 0$.

Solution $X = -B^{-1}A$ uniquely.

Problem 2) (10 points)

Find a basis for the kernel and a basis for the image of the linear transformation from \mathbb{R}^4 to \mathbb{R}^3 given by the matrix

$$A = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 2 & 1 & 2 \\ 2 & 4 & -1 & 1 \end{bmatrix}.$$

Solution Swap the first and second row to get

$$\begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 0 & 1 & 1 \\ 2 & 4 & -1 & 1 \end{bmatrix}.$$

Subtract 2 times the first row from the third row to get

$$\begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -3 & -3 \end{bmatrix}.$$

Subtract the second row from the first row and add 3 times the second row to the third row to get

$$\text{rref}(A) = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since the leading 1's sit on the first and the third columns of $\text{rref}(A)$, we can choose the first and second column vectors of A to form a basis of the image of A . Thus a basis of $\text{im}(A)$ consists of the two column vectors

$$\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}.$$

To solve the equation $A\vec{x} = 0$, since the leading 1's sit on the first and the third columns of $\text{rref}(A)$, we can choose the two components x_2 and x_4 corresponding to the columns of $\text{rref}(A)$ without leading 1's as the free variables and set $x_2 = s$ and $x_4 = t$. The two equations from $\text{rref}(A)$ corresponding to the two rows with leading 1's are

$$\begin{cases} x_1 + 2x_2 + x_4 = 0 \\ x_3 + x_4 = 0. \end{cases}$$

Thus

$$\begin{cases} x_1 = -2x_2 - x_4 = -2s - t \\ x_3 = -x_4 = -t \end{cases}$$

and

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2s - t \\ s \\ -t \\ t \end{bmatrix} = s \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix}.$$

A basis of $\ker(A)$ consists of the two column vectors

$$\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix}.$$

Problem 3) (10 points)

Let A be the matrix $\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$. Write down the matrix A^7 by first describing the linear transformation of the plane given by A in terms of rotation and dilation.

Solution Let T be the linear transformation T defined by A . Since T sends the first standard vector \vec{e}_1 to the first column of A which is

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \sqrt{2} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$$

and since T sends the second standard vector \vec{e}_2 to the second column of A which is

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \sqrt{2} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix},$$

it follows T is a rotation by $-\frac{\pi}{4}$ coupled with a stretching by the factor $\sqrt{2}$. We can also rewrite A as

$$\sqrt{2} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

in order to have unit column vectors.

When T is composed with itself to give a composition of 7 copies of T , the rotation in the transformation with matrix A^7 is by an angle equal to 7 times $-\frac{\pi}{4}$ or $-\frac{7\pi}{4}$, which is the same as $\frac{\pi}{4}$ when one adds 2π to the angle. For a rotation by an angle $\frac{\pi}{4}$, the vector \vec{e}_1 and \vec{e}_2 go respectively to the column vectors

$$\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \quad \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}.$$

The stretching in the linear transformation with matrix A^7 is by a factor of $(\sqrt{2})^7$. Thus

$$A^7 = (\sqrt{2})^7 \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 8 & -8 \\ 8 & 8 \end{bmatrix}.$$

Problem 4) (10 points)

Let

$$\vec{v}_1 = \begin{bmatrix} -\frac{1}{3} \\ \frac{2}{3} \\ -\frac{2}{3} \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} \frac{2}{3} \\ -\frac{1}{3} \\ -\frac{2}{3} \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} -\frac{2}{3} \\ -\frac{2}{3} \\ -\frac{1}{3} \end{bmatrix}.$$

Verify that these three vectors of \mathbb{R}^3 are of unit length and mutually orthogonal. Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be the rotation of \mathbb{R}^3 by 90° whose axis is \vec{v}_3 and which is in the counter-clockwise direction when one looks at the (\vec{v}_1, \vec{v}_2) -plane from the tip of \vec{v}_3 . Find the matrix B which represents T with respect to the basis $\vec{v}_1, \vec{v}_2, \vec{v}_3$ of \mathbb{R}^3 . Write down the matrix A which represents T with respect to the standard basis of \mathbb{R}^3 as a product of matrices and inverses of matrices.

Solution Each of $\vec{v}_1, \vec{v}_2, \vec{v}_3$ is of unit length, because

$$\left(\pm\frac{1}{3}\right)^2 + \left(\pm\frac{2}{3}\right)^2 + \left(\pm\frac{2}{3}\right)^2 = \frac{1}{9} + \frac{4}{9} + \frac{4}{9} = 1.$$

The three vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are mutually perpendicular, because

$$\begin{aligned} \vec{v}_1 \cdot \vec{v}_2 &= \left(-\frac{1}{3}\right)\left(\frac{2}{3}\right) + \left(\frac{2}{3}\right)\left(-\frac{1}{3}\right) + \left(-\frac{2}{3}\right)\left(-\frac{2}{3}\right) = \frac{-2-2+4}{9} = 0, \\ \vec{v}_1 \cdot \vec{v}_3 &= \left(-\frac{1}{3}\right)\left(-\frac{2}{3}\right) + \left(\frac{2}{3}\right)\left(-\frac{2}{3}\right) + \left(-\frac{2}{3}\right)\left(-\frac{1}{3}\right) = \frac{2-4+2}{9} = 0, \\ \vec{v}_2 \cdot \vec{v}_3 &= \left(\frac{2}{3}\right)\left(-\frac{2}{3}\right) + \left(-\frac{1}{3}\right)\left(-\frac{2}{3}\right) + \left(-\frac{2}{3}\right)\left(-\frac{1}{3}\right) = \frac{-4+2+2}{9} = 0. \end{aligned}$$

In \mathbb{R}^2 the counter-clockwise rotation by 90° in the (\vec{e}_1, \vec{e}_2) -plane sends \vec{e}_1 to \vec{e}_2 and sends \vec{e}_2 to $-\vec{e}_1$ and is therefore represented by the 2×2 matrix

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

with respect to the basis (\vec{e}_1, \vec{e}_2) . In \mathbb{R}^3 the 90° rotation with \vec{v}_3 as axis which is counter-clockwise when one looks down from the tip of \vec{v}_3 is given by the

matrix

$$\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

with respect to the basis $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$. Thus the matrix B which represents T with respect to the basis $\vec{v}_1, \vec{v}_2, \vec{v}_3$ of \mathbb{R}^3 is equal to

$$\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Let

$$S = [\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3] = \begin{bmatrix} -\frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} & -\frac{2}{3} \\ -\frac{2}{3} & -\frac{2}{3} & -\frac{1}{3} \end{bmatrix}.$$

Then $A = SBS^{-1}$. Thus

$$A = \begin{bmatrix} -\frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} & -\frac{2}{3} \\ -\frac{2}{3} & -\frac{2}{3} & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -\frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} & -\frac{2}{3} \\ -\frac{2}{3} & -\frac{2}{3} & -\frac{1}{3} \end{bmatrix}^{-1}.$$

Problem 5) (10 points)

Let

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

For each of the three matrices find scalars c_1, c_2, c_3 such that its image precisely consists of all column vectors

$$\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \in \mathbb{R}^3$$

satisfying $c_1y_1 + c_2y_2 + c_3y_3 = 0$. Which one of the three matrices has an image different from those of the other two?

Solution For A , subtract the first row from the second and the third rows in the following matrix

$$\begin{bmatrix} 1 & 1 & 1 & \vdots & y_1 & & \\ 1 & 0 & 0 & \vdots & & y_2 & \\ 1 & 1 & 1 & \vdots & & & y_3 \end{bmatrix}$$

to get

$$\begin{bmatrix} 1 & 1 & 1 & \vdots & y_1 & & \\ 0 & -1 & -1 & \vdots & -y_1 & +y_2 & \\ 0 & 0 & 0 & \vdots & -y_1 & & +y_3 \end{bmatrix}.$$

At this point we can already conclude that the system of linear equations

$$A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

is solvable for the unknowns x_1, x_2, x_3 if and only if $-y_1 + y_3 = 0$. We can, of course, continue with our unnecessary complete reduction to the reduced row-echelon form by multiplying the second row by -1 to get

$$\begin{bmatrix} 1 & 1 & 1 & \vdots & y_1 & & \\ 0 & 1 & 1 & \vdots & y_1 & -y_2 & \\ 0 & 0 & 0 & \vdots & -y_1 & & +y_3 \end{bmatrix}$$

and then subtracting the second row from the first row to get

$$\begin{bmatrix} 1 & 0 & 0 & \vdots & & y_2 & \\ 0 & 1 & 1 & \vdots & y_1 & -y_2 & \\ 0 & 0 & 0 & \vdots & -y_1 & & +y_3 \end{bmatrix}$$

from which we get the solutions $x_1 = y_2$ and $x_2 = -s + y_1 - y_2$ and $x_3 = s$ for any real value s provided that $-y_1 + y_3 = 0$, but there is no need to know the solutions x_1, x_2, x_3 as long as we know the necessary and sufficient condition $-y_1 + y_3 = 0$ for the existence of the solutions x_1, x_2, x_3 . For A the scalars c_1, c_2, c_3 are respectively $-1, 0, 1$.

For B , subtract the first row from the second and the third rows in the following matrix

$$\begin{bmatrix} 1 & 0 & 0 & \vdots & y_1 & & \\ 1 & 0 & 0 & \vdots & & y_2 & \\ 1 & 1 & 1 & \vdots & & & y_3 \end{bmatrix}$$

to get

$$\begin{bmatrix} 1 & 1 & 1 & \vdots & y_1 & & \\ 0 & 0 & 0 & \vdots & -y_1 & +y_2 & \\ 0 & 1 & 1 & \vdots & -y_1 & & +y_3 \end{bmatrix}.$$

At this point we can already conclude that the system of linear equations

$$B \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

is solvable for the unknowns x_1, x_2, x_3 if and only if $-y_1 + y_2 = 0$. Thus for B the scalars c_1, c_2, c_3 are respectively $-1, 1, 0$.

For C , subtract the first row from the second and the third rows in the following matrix

$$\begin{bmatrix} 1 & 0 & 1 & \vdots & y_1 & & \\ 1 & 1 & 1 & \vdots & & y_2 & \\ 1 & 0 & 1 & \vdots & & & y_3 \end{bmatrix}$$

to get

$$\begin{bmatrix} 1 & 0 & 1 & \vdots & y_1 & & \\ 0 & 1 & 0 & \vdots & -y_1 & +y_2 & \\ 0 & 0 & 0 & \vdots & -y_1 & & +y_3 \end{bmatrix}.$$

At this point we can already conclude that the system of linear equations

$$C \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

is solvable for the unknowns x_1, x_2, x_3 if and only if $-y_1 + y_3 = 0$. Thus for C the scalars c_1, c_2, c_3 are respectively $-1, 0, 1$.

The matrix B has an image different from those of A and C .

Problem 6) (10 points)

A matrix A is of the form

$$\begin{bmatrix} 0 & * & -1 & * \\ * & * & * & * \end{bmatrix}$$

where the asterisks (*) represent unknown and possibly different real numbers. Assume that A is in *reduced row-echelon form*. Find all possible such matrices A and explain why there are no other possibilities. (You may leave asterisks in your answers if each asterisk is allowed to assume any real value.) For each of the possibilities, determine its rank, its nullity, and its image.

Solution

 Let

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{bmatrix} = \begin{bmatrix} 0 & * & -1 & * \\ * & * & * & * \end{bmatrix}.$$

Since $a_{11} = 0$, we must have $a_{21} = 0$, otherwise a_{21} would be a leading 1 and the row above it would have to contain an entry 1 to the left, which is not the case.

Since the first row is not identically zero, its first nonzero coefficient must be 1 and therefore a_{11} must be a leading 1 and have value 1.

Since all other elements in the same column of a leading 1 must be zero, we conclude that $a_{22} = 0$.

The element a_{23} must be 0, otherwise it would be a leading 1 for the second row and the element a_{13} above it would have to be 0, which is not the case.

For a_{24} there are two choices, either zero or nonzero. If a_{24} is nonzero, it must be a leading 1 and the element a_{14} above it must be 0. If a_{24} is zero, then a_{14} can be any real value and we leave an asterisk in its position. So there are only the following two cases for the matrix $A = \text{rref}(A)$.

$$\begin{bmatrix} 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & -1 & * \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

For the first case

$$A = \begin{bmatrix} 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

the rank of A is 2, the nullity of A is 2 and the image of A is spanned by the two column vectors

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

For the second case

$$A = \begin{bmatrix} 0 & 1 & -1 & * \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

the rank of A is 1, the nullity of A is 3 and the image of A is spanned by the single column vector

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Problem 7) (10 points)

Consider the system of linear equations

$$\begin{cases} 3x_3 - 2x_2 = -3, \\ x_3 + 2x_4 + 2x_1 = 1, \\ x_1 + x_2 - x_3 + x_4 = 2. \end{cases}$$

(a) Write down the coefficient matrix and augmented matrix for this system.

Solution The coefficient matrix is

$$\begin{bmatrix} 0 & -2 & 3 & 0 \\ 2 & 0 & 1 & 2 \\ 1 & 1 & -1 & 1 \end{bmatrix}$$

and the augmented matrix is

$$\begin{bmatrix} 0 & -2 & 3 & 0 & \vdots & -3 \\ 2 & 0 & 1 & 2 & \vdots & 1 \\ 1 & 1 & -1 & 1 & \vdots & 2 \end{bmatrix}.$$

(b) Transform the augmented matrix by row operations so that the coefficient matrix is in reduced row-echelon form.

Solution Swap the first and the third row in the augmented matrix to get

$$\begin{bmatrix} 1 & 1 & -1 & 1 & \vdots & 2 \\ 2 & 0 & 1 & 2 & \vdots & 1 \\ 0 & -2 & 3 & 0 & \vdots & -3 \end{bmatrix}.$$

Subtract 2 times the first row from the second row in the augmented matrix to get

$$\begin{bmatrix} 1 & 1 & -1 & 1 & \vdots & 2 \\ 0 & -2 & 3 & 0 & \vdots & -3 \\ 0 & -2 & 3 & 0 & \vdots & -3 \end{bmatrix}.$$

Multiply the second row by $-\frac{1}{2}$ to get

$$\begin{bmatrix} 1 & 1 & -1 & 1 & \vdots & 2 \\ 0 & 1 & -\frac{3}{2} & 0 & \vdots & \frac{3}{2} \\ 0 & -2 & 3 & 0 & \vdots & -3 \end{bmatrix}.$$

Subtract the second row from the first row and add 2 times the second row to the third row to get

$$\begin{bmatrix} 1 & 0 & \frac{1}{2} & 1 & \vdots & \frac{1}{2} \\ 0 & 1 & -\frac{3}{2} & 0 & \vdots & \frac{3}{2} \\ 0 & 0 & 0 & 0 & \vdots & 0 \end{bmatrix},$$

where the coefficient matrix is in reduced row-echelon form.

(c) Find the general solution of the linear system.

Solution In the above reduced row-echelon form of the coefficient matrix the two leading 1's are in the first and second columns. The variables x_3 and x_4 in the other columns are free variables and we set $x_3 = s$ and $x_4 = t$. We read off from the reduced row-echelon form of the coefficient matrix

$$\begin{cases} x_1 + \frac{1}{2}x_3 + x_4 = \frac{1}{2} \\ x_2 - \frac{3}{2}x_3 = \frac{3}{2}. \end{cases}$$

Thus the general solution is given by

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}s - t + \frac{1}{2} \\ \frac{3}{2}s + \frac{3}{2} \\ s \\ t \end{bmatrix} = s \begin{bmatrix} -\frac{1}{2} \\ \frac{3}{2} \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} \frac{1}{2} \\ \frac{3}{2} \\ 0 \\ 0 \end{bmatrix}$$

for any real numbers s and t .

Problem 8) (10 points)

Let

$$A = \begin{bmatrix} 4 & 3 \\ 3 & 2 \end{bmatrix} \quad \text{and} \quad AB = \begin{bmatrix} 7 & 7 \\ 9 & 10 \end{bmatrix}.$$

Find the matrix B .

Solution We use the formula

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

to get

$$A^{-1} = \frac{1}{-1} \begin{bmatrix} 2 & -3 \\ -3 & 4 \end{bmatrix} = \begin{bmatrix} -2 & 3 \\ 3 & -4 \end{bmatrix}.$$

We get B from

$$\begin{aligned} B &= A^{-1}(AB) = \begin{bmatrix} -2 & 3 \\ 3 & -4 \end{bmatrix} \begin{bmatrix} 7 & 7 \\ 9 & 10 \end{bmatrix} \\ &= \begin{bmatrix} -14 + 27 & -14 + 30 \\ 21 - 36 & 21 - 40 \end{bmatrix} = \begin{bmatrix} 13 & 16 \\ -15 & -19 \end{bmatrix}. \end{aligned}$$

Problem 9) (10 points)

Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be the orthogonal projection onto the subspace of \mathbb{R}^3 spanned by $\begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$.

(a) Calculate $T\vec{e}_1, T\vec{e}_2, T\vec{e}_3$.

Solution The unit vector \vec{u} in the direction of $\begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$ is given by $\vec{u} = \frac{1}{3} \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$.

The orthogonal projection $T\vec{x}$ of \vec{x} onto the span of \vec{u} is given by

$$T\vec{x} = (\vec{x} \cdot \vec{u}) \vec{u} = \left(\frac{2x_1}{3} - \frac{x_2}{3} + \frac{2x_3}{3} \right) \frac{1}{3} \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}.$$

Evaluating at $\vec{x} = \vec{e}_1, \vec{e}_2, \vec{e}_3$ gives us

$$\begin{aligned} T\vec{e}_1 &= \frac{2}{3} \frac{1}{3} \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{4}{9} \\ -\frac{2}{9} \\ \frac{4}{9} \end{bmatrix}, \\ T\vec{e}_2 &= -\frac{1}{3} \frac{1}{3} \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -\frac{2}{9} \\ \frac{1}{9} \\ -\frac{2}{9} \end{bmatrix}, \\ T\vec{e}_3 &= \frac{2}{3} \frac{1}{3} \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{4}{9} \\ -\frac{2}{9} \\ \frac{4}{9} \end{bmatrix}, \end{aligned}$$

(b) Find the matrix for T .

Solution

$$T = [T\vec{e}_1 \ T\vec{e}_2 \ T\vec{e}_3] = \begin{bmatrix} \frac{4}{9} & -\frac{2}{9} & \frac{4}{9} \\ -\frac{2}{9} & \frac{1}{9} & -\frac{2}{9} \\ \frac{4}{9} & -\frac{2}{9} & \frac{4}{9} \end{bmatrix}.$$

(c) What are the dimensions of $\ker(T)$ and $\text{im}(T)$? Find bases for these two subspaces.

Solution The dimension of $\ker(T)$ is 2 and the dimension of $\text{im}(T)$ is 1. A basis of the kernel consists of the two column vectors

$$\vec{e}_1 - T\vec{e}_1, \quad \vec{e}_3 - T\vec{e}_3$$

which are

$$\begin{bmatrix} \frac{5}{9} \\ \frac{2}{9} \\ -\frac{4}{9} \end{bmatrix}, \quad \begin{bmatrix} -\frac{4}{9} \\ \frac{2}{9} \\ \frac{5}{9} \end{bmatrix}.$$

A basis of $\text{im}(T)$ consists of the single column vector $\begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$.