Solution of Second Midterm of Math 21b, November 28, 2007

Name:

MWF10	Leila Khatami
MWF11	Evan Bullock
MWF12	Yum-Tong Siu

- Start by writing your name in the above box and check your section in the box to the left.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or un-staple the packet.
- Please write neatly. Answers which are illegible for the grader cannot be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
Total:		100

Problem 1) TF questions (20 points) No justifications needed

1) T ✓F Let A and B be two $n \times n$ matrices with the same characteristic polynomial. If A is diagonalizable, then B is diagonalizable.

Solution $A = I_2$ and $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ have the same characteristic polynomial $(1-\lambda)^2$ and $A = I_2$ is diagonalizable, but B is not diagonalizable, because the kernel of $B - I_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is spanned by $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and has only dimension 1.

2) ✓T F If $\vec{v}$ is an eigenvector for a square matrix A and if $A^2 - A$ is invertible, then $\vec{v}$ is also an eigenvector for the matrix $(A^2 - A)^{-1}$.

Solution From $A\vec{v} = \lambda\vec{v}$ it follows that $(A^2 - A)\,\vec{v} = (\lambda^2 - \lambda)\,\vec{v}$ and $\vec{v} = (\lambda^2 - \lambda)\,(A^2 - A)^{-1}\,\vec{v}$. Since $\vec{v} \neq \vec{0}$, it follows that $\lambda^2 - \lambda$ cannot be 0. Thus $(A^2 - A)^{-1}\,\vec{v} = (\lambda^2 - \lambda)^{-1}\,\vec{v}$.

3) T ✓F An orthogonal matrix must be either symmetric $A^T = A$ or skew-symmetric $A^T = -A$.

Solution The orthogonal matrix $\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$, which represents the rotation of $\mathbb{R}^2$ by $\frac{\pi}{4}$ radians, is neither symmetric nor skew-symmetric.

4) T ✓F The eigenvalues of a matrix A do not change under Gauss-Jordan row operations.

Solution All the eigenvalues of I_n are equal to 1, but multiplication of the first row of I_n by 2 results in a matrix with one eigenvalue equal to 2 and the other eigenvalues equal to 1.

5) T ✓F The eigenvectors of a matrix A do not change under Gauss-Jordan row operations.

Solution The two standard vectors $\vec{e}_1$, $\vec{e}_2$ are eigenvectors for I_2, but adding the second row of I_2 to the first row of I_2 yields the matrix $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ which is not diagonalizable.

6) ✓T F If A is the matrix of an orthogonal projection onto a line in $\mathbb{R}^3$, then $\det(A - I_3) = 0$.

Solution Any nonzero vector $\vec{v}$ on the line is an eigenvector of A for the eigenvalue 1. Hence $\vec{v}$ is an eigenvector of $A - I_3$ for the eigenvalue 0. The matrix $A - I_3$ has a zero eigenvalue and as a consequence its determinant $\det(A - I_3)$ must be zero.

7) ✓T F If V is a plane in $\mathbb{R}^3$ and A is the 3×3 matrix of the reflection with respect to V, then $\det(A + I_3) = 0$.

Solution Any nonzero vector $\vec{v}$ orthogonal to the plane V is an eigenvector of A for the eigenvalue -1. Hence $\vec{v}$ is an eigenvector of $A + I_3$ for the eigenvalue 0. The matrix $A + I_3$ has a zero eigenvalue and as a consequence its determinant $\det(A + I_3)$ must be zero.

8) ✓T F The determinant of an $n \times n$ matrix A with n distinct real eigenvalues is always equal to the product of all the roots of its characteristic equation $\det(A - \lambda I) = 0$.

9) ✓T F The matrix $A = \begin{bmatrix} 2 & 2 \\ 0 & 4 \end{bmatrix}$ is similar to the matrix $B = \begin{bmatrix} 4 & 3 \\ 0 & 2 \end{bmatrix}$.

Solution Both matrices have the same eigenvalues 2 and 4, which are distinct. Hence both are diagonalizable and are similar to the same diagonal matrix. Thus they are similar.

10) T ✓F If two 2×2 matrices A and B have the same trace and determinant, then they are similar.

Solution $A = I_2$ and $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ have the trace and determinant, but they are not similar, because $A = I_2$ is diagonalizable whereas B is not.

11) ✓T F For any square matrix A we have $\det(A^9) = \left(\det(A^T)\right)^9$.

Solution Because the determinant of a matrix equals the determinant of its transpose and the determinant of the product of two matrices equals to the product of their determinants.

12) T ✓F For any 2×2 matrices A and B, AB is similar to BA.

Solution For $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $AB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $BA = 0$ are not similar.

13) ✓T F A square matrix A is invertible exactly when A^TA is invertible.

Solution If a nonzero vector $\vec{x}$ satisfies $A^TA\vec{x} = \vec{0}$, then

$$\|A\vec{x}\|^2 = (A\vec{x})^T(A\vec{x}) = \vec{x}^TA^TA\vec{x} = \vec{0}$$

and $A\vec{x} = 0$. On the other hand, $A\vec{x} = \vec{0}$ implies $A^TA\vec{x} = \vec{0}$.

14) ✓T F If $\vec{x}$ is a solution of the system of linear equations $A\vec{x} = \vec{b}$, then $\vec{x}$ is a least-squares solution of $A\vec{x} = \vec{b}$.

Solution When $A\vec{x} = \vec{b}$, one has $\left\|\vec{b} - A\vec{x}\right\| = 0$ and $\left\|\vec{b} - A\vec{x}\right\| \leq \left\|\vec{b} - A\vec{v}\right\|$ for all vectors $\vec{v}$.

15) ✓T F It is possible that for some invertible matrix A and some vector $\vec{v}$ both the length of $A^n\vec{v}$ and the length of $A^{-n}\vec{v}$ grow exponentially in the sense that there exist $C > 0$ and $a > 1$ such that $\|A^n\vec{v}\| \geq Ca^n$ and $\|A^{-n}\vec{v}\| \geq Ca^n$ for all n.

Solution This is the case when $A = \begin{bmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

16) $\boxed{\checkmark\text{T}}$ $\boxed{\text{F}}$ Any real 2×2 matrix whose eigenvalues are non-real and have modulus 1 is similar to a rotation matrix.

17) $\boxed{\checkmark\text{T}}$ $\boxed{\text{F}}$ If an $n \times n$ matrix A is diagaonalizable and λ is an eigenvalue of A, then both $A - \lambda I_n$ and $A + \lambda I_n$ are diagonalizable.

$\boxed{\text{Solution}}$ Since A is diagonalizable, there exist some invertible $n \times n$ matrix S and some diagonal matrix D such that $A = SDS^{-1}$. Then $A - \lambda I_n = S(D - \lambda I_n) S^{-1}$ and $A + \lambda I_n = S(D + \lambda I_n) S^{-1}$ and both $D - \lambda I_n$ and $D + \lambda I_n$ are diagonalizable.

18) $\boxed{\text{T}}$ $\boxed{\checkmark\text{F}}$ Every upper triangular matrix can be diagonalized.

$\boxed{\text{Solution}}$ The upper triangular matrix $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is not diagonalizable.

19) $\boxed{\text{T}}$ $\boxed{\checkmark\text{F}}$ For any matrix A the orthogonal complement of the image of A is equal to the kernel of A.

$\boxed{\text{Solution}}$ Both the image and the kernel of the matrix $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ are spanned by $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and are not orthogonal to each other.

20) $\boxed{\text{T}}$ $\boxed{\checkmark\text{F}}$ If an orthogonal matrix Q is symmetric, then Q is diagonal.

$\boxed{\text{Solution}}$ The matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is orthogonal and symmetric but not diagonal.

Problem 2) (10 points)

Write down the QR decomposition of the matrix

$$A = \begin{bmatrix} 4 & 6 \\ 2 & 7 \\ 5 & 3 \\ 2 & -2 \end{bmatrix}$$

(so that $A = QR$, where Q is a matrix whose column vectors are orthonormal and R is an upper triangular matrix with positive diagonal entries).

Solution

Denote the two column vectors by $\vec{v}_1$ and $\vec{v}_2$. Then

$$\|\vec{v}_1\|^2 = 4^2 + 2^2 + 5^2 + 2^2 = 16 + 4 + 25 + 4 = 49$$

and $\|\vec{v}_1\| = 7$. Let $\vec{u}_1$ be the unit vector parallel to $\vec{v}_1$ given by

$$\vec{u}_1 = \frac{1}{\|\vec{u}_1\|}\vec{v}_1 = \frac{1}{7}\begin{bmatrix} 4 \\ 2 \\ 5 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{4}{7} \\ \frac{2}{7} \\ \frac{5}{7} \\ \frac{2}{7} \end{bmatrix}.$$

Let V_1 be the subspace in $\mathbb{R}^4$ spanned by $\vec{u}_1$ and let $\vec{v}_2 = \vec{v}_2^{\parallel} + \vec{v}_2^{\perp}$ be the decomposition of $\vec{v}_2$ into the component $\vec{v}_2^{\parallel}$ in V_1 and the component $\vec{v}_2^{\perp}$ in the orthogonal complement $V_1^{\perp}$ of V_1. Then

$$\vec{v}_2^{\perp} = \vec{v}_2 - (\vec{u}_1 \cdot \vec{v}_2)\,\vec{u}_1$$

$$= \begin{bmatrix} 6 \\ 7 \\ 3 \\ -2 \end{bmatrix} - \left(\frac{4}{7}\cdot 6 + \frac{2}{7}\cdot 7 + \frac{5}{7}\cdot 3 + \frac{2}{7}\cdot(-2)\right)\begin{bmatrix} \frac{4}{7} \\ \frac{2}{7} \\ \frac{5}{7} \\ \frac{2}{7} \end{bmatrix}$$

$$= \begin{bmatrix} 6 \\ 7 \\ 3 \\ -2 \end{bmatrix} - \frac{4 \times 6 + 2 \times 7 + 5 \times 3 + 2 \times (-2)}{7} \begin{bmatrix} \frac{4}{7} \\ \frac{2}{7} \\ \frac{5}{7} \\ \frac{2}{7} \end{bmatrix}$$

$$= \begin{bmatrix} 6 \\ 7 \\ 3 \\ -2 \end{bmatrix} - \frac{49}{7} \begin{bmatrix} \frac{4}{7} \\ \frac{2}{7} \\ \frac{5}{7} \\ \frac{2}{7} \end{bmatrix} = \begin{bmatrix} 6 \\ 7 \\ 3 \\ -2 \end{bmatrix} - \begin{bmatrix} 4 \\ 2 \\ 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ -2 \\ -4 \end{bmatrix}$$

Since $\left\|\vec{v}_2^{\perp}\right\|^2 = 2^2 + 5^2 + (-2)^2 + 4^2 = 4 + 25 + 4 + 16 = 49$, it follows that $\left\|\vec{v}_2^{\perp}\right\| = 7$ and

$$\vec{u}_2 = \frac{1}{\left\|\vec{v}_2^{\perp}\right\|} \vec{v}_2^{\perp} = \frac{1}{7} \begin{bmatrix} 2 \\ 5 \\ -2 \\ -4 \end{bmatrix} = \begin{bmatrix} \frac{2}{7} \\ \frac{5}{7} \\ \frac{-2}{7} \\ \frac{-4}{7} \end{bmatrix}$$

is the unit vector in the direction of $\vec{v}_2^{\perp}$. Thus

$$Q = [\vec{u}_1 \quad \vec{u}_2] = \begin{bmatrix} \frac{4}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{5}{7} \\ \frac{5}{7} & \frac{-2}{7} \\ \frac{2}{7} & \frac{-4}{7} \end{bmatrix}$$

and

$$R = \begin{bmatrix} \|\vec{v}_1\| & \vec{u}_1 \cdot \vec{v}_2 \\ 0 & \left\|\vec{v}_2^{\perp}\right\| \end{bmatrix} = \begin{bmatrix} 7 & 7 \\ 0 & 7 \end{bmatrix}$$

in the QR-decomposition $A = QR$ of the given matrix A.

Problem 3) (10 points)

Find the matrix of the orthogonal projection onto the subspace of $\mathbb{R}^4$ spanned by the two vectors

$$\begin{bmatrix} 3 \\ 2 \\ 4 \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 2 \\ 1 \\ 3 \\ 0 \end{bmatrix}.$$

Solution

Recall that, for a given $n \times m$ matrix A with linearly independent column vectors, the matrix of the orthogonal projection onto the subspace spanned by the column vectors of A is equal to $A\left(A^T A\right)^{-1} A^T$. Now

$$A = \begin{bmatrix} 3 & 2 \\ 2 & 1 \\ 4 & 3 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad A^T = \begin{bmatrix} 3 & 2 & 4 & 1 \\ 2 & 1 & 3 & 0 \end{bmatrix}$$

and

$$A^T A = \begin{bmatrix} 3 & 2 & 4 & 1 \\ 2 & 1 & 3 & 0 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 2 & 1 \\ 4 & 3 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 30 & 20 \\ 20 & 14 \end{bmatrix}$$

and

$$\left(A^T A\right)^{-1} = \frac{1}{14 \times 30 - 20 \times 20} \begin{bmatrix} 14 & -20 \\ -20 & 30 \end{bmatrix}$$

$$= \frac{1}{20} \begin{bmatrix} 14 & -20 \\ -20 & 30 \end{bmatrix} = \frac{1}{10} \begin{bmatrix} 7 & -10 \\ -10 & 15 \end{bmatrix}.$$

Thus the matrix of the orthogonal projection onto the subspace of $\mathbb{R}^4$ spanned by the two vectors

$$\begin{bmatrix} 3 \\ 2 \\ 4 \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 2 \\ 1 \\ 3 \\ 0 \end{bmatrix}$$

is equal to

$$\begin{aligned}
A\left(A^T A\right)^{-1} A^T &= \begin{bmatrix} 3 & 2 \\ 2 & 1 \\ 4 & 3 \\ 1 & 0 \end{bmatrix} \frac{1}{10} \begin{bmatrix} 7 & -10 \\ -10 & 15 \end{bmatrix} \begin{bmatrix} 3 & 2 & 4 & 1 \\ 2 & 1 & 3 & 0 \end{bmatrix} \\
&= \frac{1}{10} \begin{bmatrix} 1 & 0 \\ 4 & -5 \\ -2 & 5 \\ 7 & -10 \end{bmatrix} \begin{bmatrix} 3 & 2 & 4 & 1 \\ 2 & 1 & 3 & 0 \end{bmatrix} \\
&= \frac{1}{10} \begin{bmatrix} 3 & 2 & 4 & 1 \\ 2 & 3 & 1 & 4 \\ 4 & 1 & 7 & -2 \\ 1 & 4 & -2 & 7 \end{bmatrix} \\
&= \begin{bmatrix} \frac{3}{10} & \frac{1}{5} & \frac{2}{5} & \frac{1}{10} \\ \frac{1}{5} & \frac{3}{10} & \frac{1}{10} & \frac{2}{5} \\ \frac{2}{5} & \frac{1}{10} & \frac{7}{10} & \frac{-1}{5} \\ \frac{1}{10} & \frac{2}{5} & \frac{-1}{5} & \frac{7}{10} \end{bmatrix}.
\end{aligned}$$

Problem 4) (10 points)

Let $A = \begin{bmatrix} 0 & 0 & 3 \\ 0 & 2 & 0 \\ 3 & 0 & 0 \end{bmatrix}$.

(a) Find the eigenvalues and the corresponding eigenvectors of A.

(b) Verify that there is an eigenbasis for A consisting of orthonormal vectors.

(c) For any positive integer t find the solution $\vec{x}(t)$ of the discrete linear dynamical system $\vec{x}(t+1) = A\vec{x}(t)$ with the initial value

$$\vec{x}(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Solution

(a) The characteristic equation of A is given by

$$0 = \det \begin{bmatrix} -\lambda & 0 & 3 \\ 0 & 2-\lambda & 0 \\ 3 & 0 & -\lambda \end{bmatrix}$$

which when expanded down the second column becomes

$$0 = (2-\lambda)\det \begin{bmatrix} -\lambda & 3 \\ 3 & -\lambda \end{bmatrix} = (2-\lambda)\left((-\lambda)^2 - 9\right) = (2-\lambda)(\lambda-3)(\lambda+3)$$

and we conclude that the eigenvalues of A are $2, 3, -3$. The eigenvector for the eigenvalue $\lambda_1 = 2$ is obtained from the kernel of

$$A - 2I_2 = \begin{bmatrix} -2 & 0 & 3 \\ 0 & 0 & 0 \\ 3 & 0 & -2 \end{bmatrix}$$

which is spanned by the column vector

$$\vec{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

whose length is 1. The eigenvector for the eigenvalue $\lambda_2 = 3$ is obtained from the kernel of

$$A - 3I_2 = \begin{bmatrix} -3 & 0 & 3 \\ 0 & -1 & 0 \\ 3 & 0 & -3 \end{bmatrix}$$

which is spanned by the column vector

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

and the unit vector in the same direction is

$$\vec{v}_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}.$$

The eigenvector for the eigenvalue $\lambda_3 = -3$ is obtained from the kernel of

$$A + 3I_2 = \begin{bmatrix} 3 & 0 & 3 \\ 0 & 5 & 0 \\ 3 & 0 & 3 \end{bmatrix}$$

which is spanned by the column vector

$$\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

and the unit vector in the same direction is

$$\vec{v}_3 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{-1}{\sqrt{2}} \end{bmatrix}.$$

(b) Since the dot products

$$\begin{aligned}
\vec{v}_1 \cdot \vec{v}_2 &= 0 \times \frac{1}{\sqrt{2}} + 1 \times 0 + 0 \times \frac{1}{\sqrt{2}} = 0 \\
\vec{v}_1 \cdot \vec{v}_3 &= 0 \times \frac{1}{\sqrt{2}} + 1 \times 0 + 0 \times \frac{-1}{\sqrt{2}} = 0 \\
\vec{v}_2 \cdot \vec{v}_3 &= \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}} + 0 \times 0 + \frac{1}{\sqrt{2}} \times \frac{-1}{\sqrt{2}} = 0
\end{aligned}$$

are all zero, the eigenvectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are orthonormal. Thus we have an eigenbasis $\vec{v}_1, \vec{v}_2, \vec{v}_3$ consisting of orthonormal vectors.

(c) Let

$$S = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}.$$

Then S is orthogonal and its inverse is equal to its transpose. Thus

$$\begin{aligned}
A = S \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -3 \end{bmatrix} S^{-1} &= S \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -3 \end{bmatrix} S^T \\
&= \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \end{bmatrix}
\end{aligned}$$

and for any positive integer t

$$A^t = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 2^t & 0 & 0 \\ 0 & 3^t & 0 \\ 0 & 0 & (-3)^t \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \end{bmatrix}$$

and

$$\begin{aligned} \vec{x}(t) = A^t\vec{x}(0) &= \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 2^t & 0 & 0 \\ 0 & 3^t & 0 \\ 0 & 0 & (-3)^t \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 2^t & 0 & 0 \\ 0 & 3^t & 0 \\ 0 & 0 & (-3)^t \end{bmatrix} \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \\ &= \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 0 \\ \frac{3^t}{\sqrt{2}} \\ \frac{(-3)^t}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{3^t}{2} + \frac{(-3)^t}{2} \\ 0 \\ \frac{3^t}{2} - \frac{(-3)^t}{2} \end{bmatrix} \\ &= \frac{3^t}{2} \begin{bmatrix} 1 + (-1)^t \\ 0 \\ 1 + (-1)^{t+1} \end{bmatrix}. \end{aligned}$$

Problem 5) (10 points)

Find the quadratic function $f(x) = a + bx + cx^2$ that best fits the points $(-1,-2), (-1,-1), (0,0), (1,-1), (1,-2)$ in the sense of least squares.

Solution

We attempt to solve the (inconsistent) system of equations

$$\begin{cases} a - b + c = -2 \\ a - b + c = -1 \\ a = 0 \\ a + b + c = -1 \\ a + b + c = -2 \end{cases}$$

When we put it in the form $A\vec{x} = \vec{b}$, we have

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} -2 \\ -1 \\ 0 \\ -1 \\ -2 \end{bmatrix}.$$

The least-squares solution

$$\vec{x}^* = \begin{bmatrix} a^* \\ b^* \\ c^* \end{bmatrix}$$

is given by

$$\vec{x}^* = \left(A^T A\right)^{-1} A^T \vec{b}.$$

We calculate $A^T A$ and get

$$A^T A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 4 \\ 0 & 4 & 0 \\ 4 & 0 & 4 \end{bmatrix}.$$

To find the inverse of $A^T A$ we reduce the left half of the following matrix to reduced row-echelon form

$$\begin{bmatrix} A^T A \vdots I_3 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 4 & \vdots & 1 & 0 & 0 \\ 0 & 4 & 0 & \vdots & 0 & 1 & 0 \\ 4 & 0 & 4 & \vdots & 0 & 0 & 1 \end{bmatrix}.$$

Dividing the first row by 5, we get

$$\begin{bmatrix} 1 & 0 & \frac{4}{5} & \vdots & \frac{1}{5} & 0 & 0 \\ 0 & 4 & 0 & \vdots & 0 & 1 & 0 \\ 4 & 0 & 4 & \vdots & 0 & 0 & 1 \end{bmatrix}.$$

Subtracting 4 times the first row from the third row, we get

$$\begin{bmatrix} 1 & 0 & \frac{4}{5} & \vdots & \frac{1}{5} & 0 & 0 \\ 0 & 4 & 0 & \vdots & 0 & 1 & 0 \\ 0 & 0 & \frac{4}{5} & \vdots & \frac{-4}{5} & 0 & 1 \end{bmatrix}.$$

Dividing the second row by 4, we get

$$\begin{bmatrix} 1 & 0 & \frac{4}{5} & \vdots & \frac{1}{5} & 0 & 0 \\ 0 & 1 & 0 & \vdots & 0 & \frac{1}{4} & 0 \\ 0 & 0 & \frac{4}{5} & \vdots & \frac{-4}{5} & 0 & 1 \end{bmatrix}.$$

Multiplying the third row by $\frac{5}{4}$, we get

$$\begin{bmatrix} 1 & 0 & \frac{4}{5} & \vdots & \frac{1}{5} & 0 & 0 \\ 0 & 1 & 0 & \vdots & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 1 & \vdots & -1 & 0 & \frac{5}{4} \end{bmatrix}.$$

Subtracting $\frac{4}{5}$ times the third row from the first row, we get

$$\begin{bmatrix} 1 & 0 & 0 & \vdots & 1 & 0 & -1 \\ 0 & 1 & 0 & \vdots & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 1 & \vdots & -1 & 0 & \frac{5}{4} \end{bmatrix}.$$

We can now use the right-half of the above matrix as the inverse of $A^T A$ and get

$$\left(A^T A\right)^{-1} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & \frac{1}{4} & 0 \\ -1 & 0 & \frac{5}{4} \end{bmatrix}.$$

We compute $A^T\vec{b}$ and get

$$A^T\vec{b} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ -1 \\ 0 \\ -1 \\ -2 \end{bmatrix} = \begin{bmatrix} -6 \\ 0 \\ -6 \end{bmatrix}.$$

Finally we compute $\vec{x}^* = \left(A^T A\right)^{-1} A^T\vec{b}$ and get

$$\vec{x}^* = \left(A^T A\right)^{-1} A^T\vec{b} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & \frac{1}{4} & 0 \\ -1 & 0 & \frac{5}{4} \end{bmatrix} \begin{bmatrix} -6 \\ 0 \\ -6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \frac{-3}{2} \end{bmatrix}.$$

The quadratic function that best fits the points $(-1,-2)$, $(-1,-1)$, $(0,0)$, $(1,-1)$, $(1,-2)$ in the sense of least squares is $\frac{-3}{2}x^2$.

Problem 6) (10 points)

Calculate the determinant of the matrix

$$A = \begin{bmatrix} 2 & 0 & 1 & 0 & 3 \\ 1 & 2 & 1 & 0 & 0 \\ -4 & 0 & 2 & -1 & 0 \\ 3 & 0 & -3 & 0 & 0 \\ -3 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Solution

Expanding down the second column, we get

$$\det A = 2 \det \begin{bmatrix} 2 & 1 & 0 & 3 \\ -4 & 2 & -1 & 0 \\ 3 & -3 & 0 & 0 \\ -3 & 1 & 0 & 0 \end{bmatrix}.$$

Expanding down the third column, we get

$$\det A = 2 \det \begin{bmatrix} 2 & 1 & 3 \\ 3 & -3 & 0 \\ -3 & 1 & 0 \end{bmatrix}.$$

Expanding again down the third column, we get

$$\det A = 2 \times 3 \det \begin{bmatrix} 3 & -3 \\ -3 & 1 \end{bmatrix}$$
$$= 2 \times 3\,(3 \times 1 - (-3) \times (-3)) = -36.$$

Problem 7) (10 points)

Find the determinant of the 8×8 matrix

$$A = \begin{bmatrix} a & b & b & b & b & b & b & b \\ a & a & b & b & b & b & b & b \\ a & a & a & b & b & b & b & b \\ a & a & a & a & b & b & b & b \\ a & a & a & a & a & b & b & b \\ a & a & a & a & a & a & b & b \\ a & a & a & a & a & a & a & b \\ a & a & a & a & a & a & a & a \end{bmatrix}.$$

[*Hint:* apply appropriate row operations and consider the effect of such row operations on the determinant.]

Solution

We subtract the first row from each of the rows below it and get

$$\det A = \det \begin{bmatrix} a & b & b & b & b & b & b & b \\ 0 & a-b & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a-b & a-b & 0 & 0 & 0 & 0 & 0 \\ 0 & a-b & a-b & a-b & 0 & 0 & 0 & 0 \\ 0 & a-b & a-b & a-b & a-b & 0 & 0 & 0 \\ 0 & a-b & a-b & a-b & a-b & a-b & 0 & 0 \\ 0 & a-b & a-b & a-b & a-b & a-b & a-b & 0 \\ 0 & a-b & a-b & a-b & a-b & a-b & a-b & a-b \end{bmatrix},$$

because such row operations do not change the value of the determinant. Expanding down the first column and using the fact that the determinant of a lower triangular matrix is equal to the product of the entries on the diagonal, we get

$$\det A = a(a-b)^7.$$

Problem 8) (10 points)

Consider a discrete linear dynamical system $\vec{x}(t+1) = A\vec{x}(t)$, where A is a 3×3 matrix. Suppose that $\vec{x}(0)$, $\vec{x}(1)$, $\vec{x}(2)$ are linearly independent. Further assume that $\vec{x}(3) = a\vec{x}(0) + b\vec{x}(1)$. Let T be the linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$ which is represented by the matrix A with respect to the standard basis of $\mathbb{R}^3$.

(a) Find the matrix B which represents the linear transformation T with respect to the basis $\mathfrak{B} = \{\vec{x}(0), \vec{x}(1), \vec{x}(2)\}$, *i.e.*, find $B = [T]_{\mathfrak{B}}$ in terms of a and b.

(b) If we are further given that

$$\vec{x}(0) = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{2} \end{bmatrix}, \quad \vec{x}(1) = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}, \quad x(2) = \begin{bmatrix} 0 \\ -5 \\ 0 \end{bmatrix}, \quad \vec{x}(3) = \begin{bmatrix} 9 \\ 0 \\ 1 \end{bmatrix}.$$

Determine a and b.

(c) Find a matrix S such that $A = SBS^{-1}$.

Solution

(a) From $\vec{x}(t+1) = A\vec{x}(t)$ we conclude from $t = 0, 1, 2$ that

$$\begin{cases} \vec{x}(1) = A\vec{x}(0) \\ \vec{x}(2) = A\vec{x}(1) \\ \vec{x}(3) = A\vec{x}(2). \end{cases}$$

Since $\vec{x}(3) = a\vec{x}(0) + b\vec{x}(1)$, we can rewrite these three equations as

$$\begin{cases} A\vec{x}(0) = \vec{x}(1) \\ \vec{A}\vec{x}(1) = \vec{x}(2) \\ \vec{A}\vec{x}(2) = a\vec{x}(0) + b\vec{x}(1). \end{cases}$$

Thus the matrix B which represents the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ with respect to the basis $\mathfrak{B} = \{\vec{x}(0), \vec{x}(1), \vec{x}(2)\}$ is given by

$$B = \begin{bmatrix} 0 & 0 & a \\ 1 & 0 & b \\ 0 & 1 & 0 \end{bmatrix}.$$

(b) To determine a and b from $\vec{x}(3) = a\vec{x}(0) + b\vec{x}(1)$, we use

$$\begin{bmatrix} 9 \\ 0 \\ 1 \end{bmatrix} = a \begin{bmatrix} 0 \\ 0 \\ \frac{1}{2} \end{bmatrix} + b \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix},$$

from whose first and third components we conclude that $9 = 3b$ and $1 = \frac{1}{2}a$. Hence $a = 2$ and $b = 3$ and

$$B = \begin{bmatrix} 0 & 0 & 2 \\ 1 & 0 & 3 \\ 0 & 1 & 0 \end{bmatrix}.$$

(c) Let

$$S = \begin{bmatrix} \vec{x}(0) & \vec{x}(1) & \vec{x}(2) \end{bmatrix} = \begin{bmatrix} 0 & 3 & 0 \\ 0 & 0 & -5 \\ \frac{1}{2} & 0 & 0 \end{bmatrix}.$$

Then $A = SBS^{-1}$.

Problem 9) (10 points)

(a) Verify that any real 2×2 matrix A is the sum of a symmetric 2×2 matrix and a skew-symmetric 2×2 matrix. [*Hint*: it may be helpful to consider the sum of A and its transpose and consider the difference of A and its transpose.]

(b) Let V be the linear space of all real 2×2 matrices. Let L be the linear transformation from V to V which sends a 2×2 matrix A to $A+2A^T$, where A^T is the transpose of A. Show that any symmetric matrix A is an eigenvector of L and determine its eigenvalue. Show that any skew-symmetric matrix A is an eigenvector of L and determine its eigenvalue.

(c) Use (a) and (b) to show that there is a basis of V which is an eigenbasis for L (*i.e.*, L is diagonalizable) and write down the eigenvectors in the eigenbasis and their corresponding eigenvalues.

(d) Write down the 4×4 matrix M which represents L with respect to the basis

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

of V. Use (c) to obtain an invertible 4×4 matrix S such that $S^{-1}MS$ is a diagonal 4×4 matrix D and write down the diagonal entries of D.

Solution

(a) Any real 2×2 matrix A can be written as

$$A = \frac{1}{2}\left(A + A^T\right) + \frac{1}{2}\left(A - A^T\right).$$

The matrix $\frac{1}{2}\left(A + A^T\right)$ is symmetric and the matrix $\frac{1}{2}\left(A - A^T\right)$ is skew-symmetric. Thus any real 2×2 matrix A is the sum of a symmetric 2×2 matrix and a skew-symmetric 2×2 matrix.

(b) For any symmetric real 2×2 matrix A, we have

$$L(A) = A + 2A^T = A + 2A = 3A.$$

Thus any nonzero symmetric real 2×2 matrix A is an eigenvector for L whose eigenvalue is 3.

For any skew-symmetric real 2×2 matrix A, we have

$$L(A) = A + 2A^T = A - 2A = -A.$$

Thus any nonzero skew-symmetric real 2×2 matrix A is an eigenvector for L whose eigenvalue is -1.

(c) Since any real 2×2 matrix A is the sum of a symmetric 2×2 matrix and a skew-symmetric 2×2 matrix, it follows that there is a basis of V which is an eigenbasis for L. Explicitly, we can choose

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

as the eigenbasis whose corresponding eigenvalues are $3, 3, 3, -1$.

(d) Since

$$\begin{aligned}
L : \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} &\mapsto \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 2\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = 3\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \\
L : \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} &\mapsto \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 2\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \\
L : \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} &\mapsto \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + 2\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \\
L : \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} &\mapsto \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + 2\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = 3\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},
\end{aligned}$$

it follows that

$$M = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}.$$

We now express the eigenbasis

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

in terms of

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

and get

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$
$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},$$
$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},$$
$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Thus we can set

$$S = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

to get

$$S^{-1}MS = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

The diagonal entries of the diagonal 4×4 matrix $D = S^{-1}MS$ are $3, 3, 3, -1$ from the upper left-hand corner to the lower right-hand corner.