# PRACTICE EXAMINATION THREE
# FOR SECOND MID-TERM

| November 28, 2007 | Math 21b, Fall 2007 |
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|---|---|---|
| 1 | | 20 |
| 2 | | 10 |
| 3 | | 10 |
| 4 | | 10 |
| 5 | | 10 |
| 6 | | 10 |
| 7 | | 10 |
| 8 | | 10 |
| 9 | | 10 |
| | | |
| Total | | 100 |

- Start by writing your name in the above box and check your section in the box to the left.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or un-staple the packet.
- Please write neatly. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

Problem 1) (20 points) True or False? No justifications are needed.

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|---|---|---|---|
| 1) | T | F | A $4 \times 4$ matrix whose entries are all 4 has determinant $4^4$. |
| 2) | T | F | $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is an orthogonal matrix. |
| 3) | T | F | Every $2 \times 2$ matrix is diagonalizable over the complex numbers. |
| 4) | T | F | If $A$ is a projection onto a linear subspace $V$, then $\mathrm{im}(A) = V$ and $\ker(A) = V^\perp$. |
| 5) | T | F | If $A$ is a $3 \times 3$ matrix representing reflection about a line in $\mathbb{R}^3$, then $A$ is symmetric. |
| 6) | T | F | If $A$ is a matrix with orthonormal columns, then $\vec{x} = A^T\vec{b}$ must be a least-squares solution of the system $A\vec{x} = \vec{b}$. |
| 7) | T | F | If $A$ represents the projection onto a line in $\mathbb{R}^2$, then $A$ is similar to $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. |
| 8) | T | F | If $A$ is a square matrix such that $A\vec{v} \cdot A\vec{w} = 0$, whenever $\vec{v} \cdot \vec{w} = 0$, then $A$ is an orthogonal matrix. |
| 9) | T | F | If $A$ is a $2 \times 2$ matrix with the eigenvalues 0 and 1, then $A$ is the matrix of an orthogonal projection. |
| 10) | T | F | If $A$ and $B$ are similar, then they have the same eigenvectors. |
| 11) | T | F | If $A$ and $B$ are both diagonalizable, then $AB$ is diagonalizable. |
| 12) | T | F | If $\lambda$ is an eigenvalue of $A$ and $\mu$ is an eigenvalue of $B$, then $\lambda\mu$ is an eigenvalue of $AB$. |
| 13) | T | F | If $S^{-1}AS$ is a diagonal matrix, then the columns of $S$ must be eigenvectors of $A$. |
| 14) | T | F | If $\lambda$ is an eigenvalue of a $2 \times 2$ matrix $A$, then $-\lambda$ is an eigenvalue of $A$. |
| 15) | T | F | If $A$ is a $5 \times 5$ matrix, then $\det(5A) = 25\det(A)$. |
| 16) | T | F | Any two $2 \times 2$ matrices whose eigenvectors are equal must be similar. |
| 17) | T | F | The standard basis vectors of $\mathbf{R}^n$ are the eigenvectors of every diagonal $n \times n$ matrix. |
| 18) | T | F | If $A^2 = A$, then every eigenvalue $\lambda$ of $A$ is either $\lambda = 1$ or $\lambda = 0$. |
| 19) | T | F | If $A$ is a symmetric matrix with characteristic polynomial $(1-\lambda)^2(2-\lambda)$, then the 1-eigenspace of $A$ is the orthogonal complement of the 2-eigenspace of $A$. |
| 20) | T | F | A $2 \times 2$ matrix with characteristic polynomial $f_A(\lambda) = (2-\lambda)^2$ is similar to either $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ or $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. |

Problem 2) (10 points)

Check the boxes, where the two matrices $A$ and $B$ are similar. No explanations are necessary for this problem. Each pair counts 2 points.

a) ☐ $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

b) ☐ $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

c) ☐ $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ $B = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

d) ☐ $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$

e) ☐ $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

Problem 3) (10 points)

Consider the matrix $A = \begin{bmatrix} 5 & 1 & 1 & 1 & 1 & 1 \\ 1 & 5 & 1 & 1 & 1 & 1 \\ 1 & 1 & 5 & 1 & 1 & 1 \\ 1 & 1 & 1 & 5 & 1 & 1 \\ 1 & 1 & 1 & 1 & 5 & 1 \\ 1 & 1 & 1 & 1 & 1 & 5 \end{bmatrix}$.

a) Find the kernel of $B = A - 4I_6$.

b) Show that $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ is an eigenvector of $A$. What is the eigenvalue?

c) Find all the eigenvalues of $A$ with their algebraic multiplicities.

d) Write down the eigenbasis of $A$. What are the geometric multiplicities of the eigenvalues?

e) What is $\det(A)$?

Problem 4) (10 points)

Find the function $y = f(x) = a + b2^x$, which best fits the data

| x | y |
|---|---|
| 0 | 1 |
| 1 | 3 |
| 2 | 7 |

Problem 5) (10 points)

Let $A = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 2 & -1 \\ 6 & 0 & -4 \end{bmatrix}$. You are told that $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is an eigenbasis for $A$, where:

- $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ belongs to the eigenvalue $-1$,
- $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ belongs to the eigenvalue $2$,
- $\vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$ belongs to the eigenvalue $-2$.

Find an eigenbasis for $A^T$.

Problem 6) (10 points)

a) (4 points) Find the Gram-Schmidt orthogonalization of the basis $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$.

b) (4 points) Find the QR decomposition of the matrix $A = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix}$.

c) (2 points) What is the volume of the parallelepiped spanned by the column vectors of $A$?

Problem 7) (10 points)

a) Find the determinant of the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 3 & 4 \\ 0 & 0 & 2 & 3 & 4 \\ 0 & 0 & 3 & 4 & 5 \\ 3 & -2 & -3 & -4 & 0 \end{bmatrix}$$

b) Find the determinant of the matrix

$$\begin{bmatrix} 1 & 2 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 3 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 0 & 2 & 3 \end{bmatrix}$$

Problem 8) (10 points)

To support their teams, Harvard and Yale fans both sell T-shirts in November 2004. Each day, the following happens:

- Every Harvard fan recruits two more Harvard fans and four Yale fans.

- Every Yale fan recruits two more Yale fans and one Harvard fan.

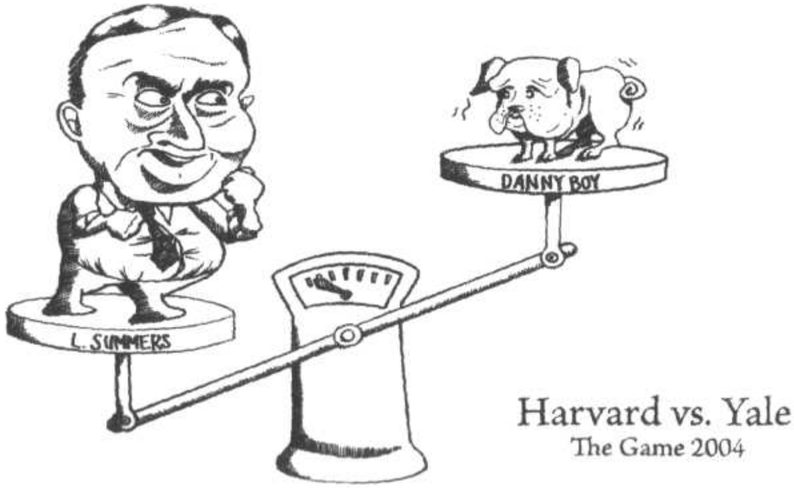
Let $H(t)$ and $Y(t)$ be the number of Harvard fans and Yale fans, respectively, on day $t$. We can model the above situation with a dynamical system

$$\begin{bmatrix} H(t+1) \\ Y(t+1) \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} H(t) \\ Y(t) \end{bmatrix}.$$

Suppose that $H(0) = Y(0) = 100$.

a) (5 points) Find a closed formula for $H(t), Y(t)$.

b) (5 points) What happens with $H(t)/Y(t)$ for large $t$?




| Problem 9) (10 points) |
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Let $V$ be the 3-dimensional subspace of $\mathbb{R}^4$ spanned by

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}.$$

Let $\{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$ be an orthonormal basis of $V$ and $A$ be the matrix $A = \begin{bmatrix} | & | & | \\ \vec{w}_1 & \vec{w}_2 & \vec{w}_3 \\ | & | & | \end{bmatrix}$. Find $\det(AA^T)$ and $\det(A^TA)$.

**Hint.** Think before calculating. You don't actually need to compute $\vec{w}_1, \vec{w}_2, \vec{w}_3$, nor do you have to multiply matrices to solve the problem.