

NOTATION. Complex numbers are written as  $z = x + iy = r \exp(i\phi) = r \cos(\phi) + ir \sin(\phi)$ . The real number  $r = |z|$  is called the **absolute value** of  $z$ , the value  $\phi$  is the **argument** and denoted by  $\arg(z)$ . Complex numbers contain the **real numbers**  $z = x + i0$  as a subset. One writes  $\operatorname{Re}(z) = x$  and  $\operatorname{Im}(z) = y$  if  $z = x + iy$ .

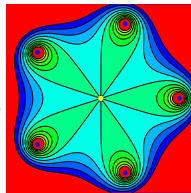
ARITHMETIC. Complex numbers are added like vectors:  $x + iy + u + iv = (x + u) + i(y + v)$  and multiplied as  $z * w = (x + iy)(u + iv) = xu - yv + i(yu - xv)$ . If  $z \neq 0$ , one can divide  $1/z = 1/(x + iy) = (x - iy)/(x^2 + y^2)$ .

ABSOLUTE VALUE AND ARGUMENT. The absolute value  $|z| = \sqrt{x^2 + y^2}$  satisfies  $|zw| = |z| |w|$ . The argument satisfies  $\arg(zw) = \arg(z) + \arg(w)$ . These are direct consequences of the polar representation  $z = r \exp(i\phi)$ ,  $w = s \exp(i\psi)$ ,  $zw = rs \exp(i(\phi + \psi))$ .

GEOMETRIC INTERPRETATION. If  $z = x + iy$  is written as a vector  $\begin{bmatrix} x \\ y \end{bmatrix}$ , then multiplication with an other complex number  $w$  is a **dilation-rotation**: a scaling by  $|w|$  and a rotation by  $\arg(w)$ .

THE DE MOIVRE FORMULA.  $z^n = \exp(in\phi) = \cos(n\phi) + i \sin(n\phi) = (\cos(\phi) + i \sin(\phi))^n$  follows directly from  $z = \exp(i\phi)$  but it is magic: it leads for example to formulas like  $\cos(3\phi) = \cos(\phi)^3 - 3 \cos(\phi) \sin^2(\phi)$  which would be more difficult to come by using geometrical or power series arguments. This formula is useful for example in integration problems like  $\int \cos(x)^3 dx$ , which can be solved by using the above deMoivre formula.

THE UNIT CIRCLE. Complex numbers of length 1 have the form  $z = \exp(i\phi)$  and are located on the **unit circle**. The characteristic polynomial  $f_A(\lambda) = \lambda^5 - 1$  of the matrix  $\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$  has all roots on the unit circle. The roots  $\exp(2\pi ki/5)$ , for  $k = 0, \dots, 4$  lie on the unit circle.



THE LOGARITHM.  $\log(z)$  is defined for  $z \neq 0$  as  $\log|z| + i \arg(z)$ . For example,  $\log(2i) = \log(2) + i\pi/2$ . Riddle: what is  $i^i$ ? ( $i^i = e^{i \log(i)} = e^{i\pi/2} = e^{-\pi/2}$ ). The logarithm is not defined at 0 and the imaginary part is define only up to  $2\pi$ . For example, both  $i\pi/2$  and  $5i\pi/2$  are equal to  $\log(i)$ .

HISTORY. The struggle with  $\sqrt{-1}$  is historically quite interesting. Nagging questions appeared for example when trying to find closed solutions for roots of polynomials. Cardano (1501-1576) was one of the mathematicians who at least considered complex numbers but called them arithmetic subtleties which were "as refined as useless". With Bombelli (1526-1573), complex numbers found some practical use. Descartes (1596-1650) called roots of negative numbers "imaginary". Although the fundamental theorem of algebra (below) was still not proved in the 18th century, and complex numbers were not fully understood, the square root of minus one  $\sqrt{-1}$  was used more and more. Euler (1707-1783) made the observation that  $\exp(ix) = \cos x + i \sin x$  which has as a special case the **magic formula**  $e^{i\pi} + 1 = 0$  which relate the constants 0, 1,  $\pi$ ,  $e$  in one equation.

For decades, many mathematicians still thought complex numbers were a **waste of time**. Others used complex numbers extensively in their work. In 1620, Girard suggested that an equation may have as many roots as its degree in 1620. Leibniz (1646-1716) spent quite a bit of time trying to apply the laws of algebra to complex numbers. He and Johann Bernoulli used imaginary numbers as integration aids. Lambert used complex numbers for map projections, d'Alembert used them in hydrodynamics, while Euler, D'Alembert and Lagrange used them in their incorrect proofs of the fundamental theorem of algebra. Euler write first the symbol  $i$  for  $\sqrt{-1}$ .

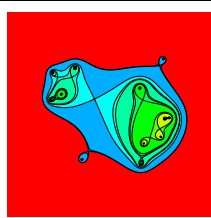
Gauss published the first correct proof of the fundamental theorem of algebra in his doctoral thesis, but still claimed in 1825 that **the true metaphysics of the square root of  $-1$  is elusive** as late as 1825. By 1831 Gauss overcame his uncertainty about complex numbers and published his work on the geometric representation of complex numbers as points in the plane. In 1797, a Norwegian Caspar Wessel (1745-1818) and in 1806 a Swiss clerk named Jean Robert Argand (1768-1822) (who stated the theorem the first time for polynomials with complex coefficients) did similar work. But these efforts went unnoticed. William Rowan Hamilton (1805-1865) (who would also discover the quaternions while walking over a bridge) expressed in 1833 complex numbers as vectors.

Complex numbers continued to develop to **complex function theory** or **chaos theory**, a branch of dynamical systems theory. Complex numbers are helpful in geometry in number theory or in quantum mechanics. Once believed fictitious they are now most "natural numbers" and the "natural numbers" themselves are in fact the most "complex". A philosopher who asks "does  $\sqrt{-1}$  really exist?" might be shown the representation of  $x + iy$  as  $\begin{bmatrix} x & -y \\ y & x \end{bmatrix}$ . When adding or multiplying such dilation-rotation matrices, they behave like complex numbers: for example  $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$  plays the role of  $i$ .

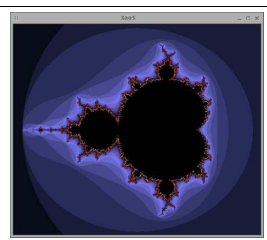
**FUNDAMENTAL THEOREM OF ALGEBRA.** (Gauss 1799) A polynomial of degree  $n$  has exactly  $n$  roots.

**CONSEQUENCE:** A  $n \times n$  MATRIX HAS  $n$  EIGENVALUES. The characteristic polynomial  $f_A(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \dots + a_1\lambda + a_0$  satisfies  $f_A(\lambda) = (\lambda - \lambda_1) \dots (\lambda - \lambda_n)$ , where  $\lambda_i$  are the roots of  $f$ .

**TRACE AND DETERMINANT.** Comparing  $f_A(\lambda) = (\lambda - \lambda_1) \dots (\lambda - \lambda_n)$  with  $\lambda^n - \text{tr}(A) \lambda^{n-1} + \dots + (-1)^n \det(A)$  gives  $\boxed{\text{tr}(A) = \lambda_1 + \dots + \lambda_n, \det(A) = \lambda_1 \dots \lambda_n.}$



**COMPLEX FUNCTIONS.** The characteristic polynomial is an example of a function  $f$  from  $\mathbb{C}$  to  $\mathbb{C}$ . The graph of this function would live in  $\mathbb{C} \times \mathbb{C}$  which corresponds to a four dimensional real space. One can visualize the function however with the real-valued function  $z \mapsto |f(z)|$ . The figure to the left shows the contour lines of such a function  $z \mapsto |f(z)|$ , where  $f$  is a polynomial.



**ITERATION OF POLYNOMIALS.** A topic which is off this course (it would be a course by itself) is the iteration of polynomials like  $f_c(z) = z^2 + c$ . The set of parameter values  $c$  for which the iterates  $f_c(0), f_c^2(0) = f_c(f_c(0)), \dots, f_c^n(0)$  stay bounded is called the **Mandelbrot set**. It is the fractal black region in the picture to the left. The now already dusty object appears everywhere, from photoshop plugins to decorations. In Mathematica, you can compute the set very quickly (see <http://www.math.harvard.edu/computing/math/mandelbrot.m>).

**COMPLEX NUMBERS IN MATHEMATICA OR MAPLE.** In both computer algebra systems, the letter  $I$  is used for  $i = \sqrt{-1}$ . In Maple, you can ask  $\log(1 + I)$ , in Mathematica, this would be  $\text{Log}[1 + I]$ . Eigenvalues or eigenvectors of a matrix will in general involve complex numbers. For example, in Mathematica,  $\text{Eigenvalues}[A]$  gives the eigenvalues of a matrix  $A$  and  $\text{Eigensystem}[A]$  gives the eigenvalues and the corresponding eigenvectors.

**EIGENVALUES AND EIGENVECTORS OF A ROTATION.** The rotation matrix  $A = \begin{bmatrix} \cos(\phi) & \sin(\phi) \\ -\sin(\phi) & \cos(\phi) \end{bmatrix}$  has the characteristic polynomial  $\lambda^2 - 2\cos(\phi) + 1$ . The eigenvalues are  $\cos(\phi) \pm \sqrt{\cos^2(\phi) - 1} = \cos(\phi) \pm i\sin(\phi) = \exp(\pm i\phi)$ . The eigenvector to  $\lambda_1 = \exp(i\phi)$  is  $v_1 = \begin{bmatrix} -i \\ 1 \end{bmatrix}$  and the eigenvector to the eigenvector  $\lambda_2 = \exp(-i\phi)$  is  $v_2 = \begin{bmatrix} i \\ 1 \end{bmatrix}$ .