FOURIER SERIES

Smooth functions $f(x)$ on $[-\pi,\pi]$ form a linear space X. There is an **inner product** in X defined by

$$\langle f,g\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)g(x)\,dx$$

It allows to define angles, length, projections etc in the space X as we did in finite dimensions.

THE FOURIER BASIS.

THEOREM. The functions $\left\{\cos(nx), \sin(nx), 1/\sqrt{2}\right\}$ form an orthonormal family in X.

Proof. To check linear independence a few integrals need to be computed. For all $n,m \geq 1$, with $n \neq m$ you have to show:

$$\langle \tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}\rangle = 1$$
$$\langle \cos(nx), \cos(nx)\rangle = 1, \langle \cos(nx), \cos(mx)\rangle = 0$$
$$\langle \sin(nx), \sin(nx)\rangle = 1, \langle \sin(nx), \sin(mx)\rangle = 0$$
$$\langle \sin(nx), \cos(mx)\rangle = 0$$
$$\langle \sin(nx), 1/\sqrt{2}\rangle = 0$$
$$\langle \cos(nx), 1/\sqrt{2}\rangle = 0$$

To verify the above integrals in the homework, the following trigonometric identities are useful:

$$\begin{aligned} 2\cos(nx)\cos(my) &= \cos(nx-my)+\cos(nx+my) \\ 2\sin(nx)\sin(my) &= \cos(nx-my)-\cos(nx+my) \\ 2\sin(nx)\cos(my) &= \sin(nx+my)+\sin(nx-my) \end{aligned}$$

FOURIER COEFFICIENTS. The **Fourier coefficients** of f are defined as

$$a_0 = \langle f, 1/\sqrt{2}\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)/\sqrt{2}\,dx$$
$$a_n = \langle f, \cos(nt)\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx$$
$$b_n = \langle f, \sin(nt)\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx$$

FOURIER SERIES. The **Fourier representation** of a smooth function f is the identity

$$f(x) = \frac{a_0}{\sqrt{2}} + \sum_{k=1}^{\infty} a_k\cos(kx) + \sum_{k=1}^{\infty} b_k\sin(kx)$$

We take it for granted that the series converges and that the identity holds for all x.

ODD AND EVEN FUNCTIONS. Here is some advise which can save time when computing Fourier series:

If f is odd: $f(x) = -f(-x)$, then f has a sin series.
If f is even: $f(x) = f(-x)$, then f has a cos series.
If you integrate an odd function over $[-\pi,\pi]$ you get 0.
The product of two odd functions is even, the product between an even and an odd function is odd.

EXAMPLE 1. Let $f(x) = x$ on $[-\pi,\pi]$. This is an odd function $(f(-x)+f(x) = 0)$ so that it has a sin series: with $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin(nx)\,dx = \frac{-1}{\pi}(x\cos(nx)/n + \sin(nx)/n^2|_{-\pi}^{\pi}) = 2(-1)^{n+1}/n$, we get $x = \sum_{n=1}^{\infty} 2\frac{(-1)^{n+1}}{n}\sin(nx)$. For example

$$\frac{\pi}{2} = 2\left(\frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7}\ldots\right)$$

is a **formula of Leibnitz**.

EXAMPLE 2. Let $f(x) = \cos(x) + 1/7\cos(5x)$. This **trigonometric polynomial** is already the Fourier series. The nonzero coefficients are $a_1 = 1, a_5 = 1/7$.

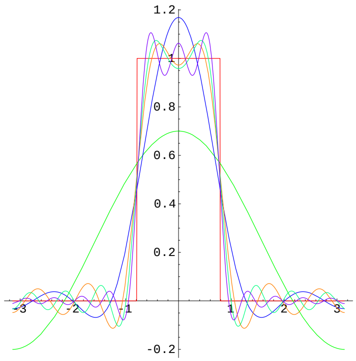
EXAMPLE 3. Let $f(x) = 1$ on $[-\pi/2, \pi/2]$ and $f(x) = 0$ else. This is an even function $f(-x) - f(x) = 0$ so that it has a cos series: with $a_0 = 1/(\sqrt{2}), a_n = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} 1\cos(nx)\, dx = \frac{\sin(nx)}{\pi n}|_{-\pi/2}^{\pi/2} = \frac{2(-1)^m}{\pi(2m+1)}$ if $n = 2m+1$ is odd and 0 else. So, the series is

$$f(x) = \frac{1}{2} + \frac{2}{\pi}(\frac{\cos(x)}{1} - \frac{\cos(3x)}{3} + \frac{\cos(5x)}{5} - ...)$$

Remark. The function in Example 3 is not smooth but Fourier theory still works. What happened at the discontinuity point $\pi/2$? The Fourier series gives 0. Diplomatically, it has chosen the point in the middle of the limits from the right and the limit from the left.

FOURIER APPROXIMATION. For a smooth function f, the Fourier series of f converges to f. The Fourier coefficients are the coordinates of f in the Fourier basis.

The function $f_n(x) = \sum_{k=1}^{n} a_k \sin(kx)$ is called a **Fourier approximation** of f. The picture to the right shows an approximation of a piecewise continuous even function in example 3).



THE PARSEVAL EQUALITY. When evaluating the square of the length of f with the square of the length of the series, we get

$$||f||^2 = a_0^2 + \sum_{k=1}^{\infty} a_k^2 + b_k^2\ .$$

EXAMPLE. We have seen in example 1 that $f(x) = x = 2(\sin(x) - \sin(2x)/2 + \sin(3x)/3 - \sin(4x)/4 + ...$ Because the Fourier coefficients are $b_k = 2(-1)^{k+1}/k$, we have $4(1 + 1/4 + 1/9 + ...) = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\, dx = 2\pi^2/3$ and so

$$\frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + ... = \frac{\pi^2}{6}$$

HOMEWORK

1. Verify that the functions $\cos(nx)$, $\sin(nx)$, $\frac{1}{\sqrt{2}}$ form an orthonormal family.

2. Find the Fourier series of the function $f(x) = |x|$.

3. Find the Fourier series of the function $\cos^2(x) + 5\sin(x) + 5$. You may find the double angle formula $\cos^2(x) = \frac{\cos(2x)+1}{2}$ useful.

4. Find the Fourier series of the function $f(x) = |\sin(x)|$.

5. In the preceding problem 4, you would have gotten a series

$$f(x) = \frac{2}{\pi} - \frac{4}{\pi}\left(\frac{\cos(2x)}{2^2-1} + \frac{\cos(4x)}{4^2-1} + \frac{\cos(6x)}{6^2-1} + \cdots\right).$$

   Use Parseval's identity to find the value of

$$\frac{1}{(2^2-1)^2} + \frac{1}{(4^2-1)^2} + \frac{1}{(6^2-1)^2} + \cdots$$