HEAT AND WAVE EQUATION

FUNCTIONS OF TWO VARIABLES. We consider functions $f(x,t)$ which are for fixed t a piecewise smooth function in x. Analogously as we studied the motion of a **vector** $\vec{v}(t)$, we are now interested in the motion of a **function** f in time t. While the governing equation for a vector was an ordinary differential equation $\dot{x} = Ax$ (ODE), the describing equation is now be a **partial differential equation** (PDE) $\dot{f} = T(f)$. The function $f(x,t)$ could denote the **temperature of a stick** at a position x at time t or the **displacement of a string** at the position x at time t. The motion of these dynamical systems will be easy to describe in the orthonormal Fourier basis $1/\sqrt{2}, \sin(nx), \cos(nx)$ treated in an earlier lecture.

PARTIAL DERIVATIVES. We write $f_x(x,t)$ and $f_t(x,t)$ for the **partial derivatives** with respect to x or t. The notation $f_{xx}(x,t)$ means that we differentiate twice with respect to x.

Example: for $f(x,t) = \cos(x+4t^2)$, we have
- $f_x(x,t) = -\sin(x+4t^2)$
- $f_t(x,t) = -8t\sin(x+4t^2)$.
- $f_{xx}(x,t) = -\cos(x+4t^2)$.

One also uses the notation $\frac{\partial f(x,y)}{\partial x}$ for the partial derivative with respect to x. Tired of all the "partial derivative signs", we always write $f_x(x,t)$ for the partial derivative with respect to x and $f_t(x,t)$ for the partial derivative with respect to t.

PARTIAL DIFFERENTIAL EQUATIONS. A partial differential equation is an equation for an unknown function $f(x,t)$ in which different partial derivatives occur.

- $f_t(x,t) + f_x(x,t) = 0$ with $f(x,0) = \sin(x)$ has a solution $f(x,t) = \sin(x-t)$.
- $f_{tt}(x,t) - f_{xx}(x,t) = 0$ with $f(x,0) = \sin(x)$ and $f_t(x,0) = 0$ has a solution $f(x,t) = (\sin(x-t) + \sin(x+t))/2$.

THE HEAT EQUATION. The temperature distribution $f(x,t)$ in a metal bar $[0,\pi]$ satisfies the **heat equation**

$$f_t(x,t) = \mu f_{xx}(x,t)$$

This partial differential equation tells that the rate of change of the temperature at x is proportional to the second space derivative of $f(x,t)$ at x. The function $f(x,t)$ is assumed to be zero at both ends of the bar and $f(x) = f(x,t)$ is a given initial temperature distribution. The constant μ depends on the heat conductivity properties of the material. Metals for example conduct heat well and would lead to a large μ.

REWRITING THE PROBLEM. We can write the problem as

$$\boxed{\frac{d}{dt} f = \mu D^2 f}$$

We will solve the problem in the same way as we solved linear differential equations:

$$\boxed{\frac{d}{dt}\vec{x} = A\vec{x}}$$

where A is a matrix - $\boxed{\text{by diagonalization}}$.

We use that the Fourier basis is just the diagonalization: $D^2\cos(nx) = -n^2\cos(nx)$ and $D^2\sin(nx) = -n^2\sin(nx)$ show that $\cos(nx)$ and $\sin(nx)$ are eigenfunctions to D^2 with eigenvalue n^2. By a symmetry trick, we can focus on sin-series from now on.

SOLVING THE HEAT EQUATION WITH FOURIER THEORY. The heat equation $f_t(x,t) = \mu f_{xx}(x,t)$ with smooth $f(x,0) = f(x)$, $f(0,t) = f(\pi,t) = 0$ has the solution

$$f(x,t) = \sum_{n=1}^{\infty} b_n \sin(nx) e^{-n^2 \mu t}$$

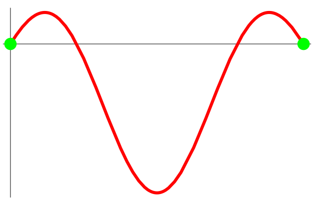
$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin(nx)\, dx$$

Proof: With the initial condition $f(x) = \sin(nx)$, we have the evolution $f(x,t) = e^{-\mu n^2 t} \sin(nx)$. If $f(x) = \sum_{n=1}^{\infty} b_n \sin(nx)$ then $f(x,t) = \sum_{n=1}^{\infty} b_n e^{-\mu n^2 t} \sin(nx)$.

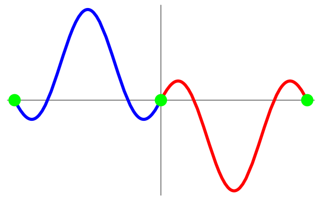
A SYMMETRY TRICK. Given a function f on the interval $[0,\pi]$ which is zero at 0 and π. It can be extended to an odd function on the doubled integral $[-\pi,\pi]$.

The Fourier series of an odd function is a pure sin-series. The Fourier coefficients are $b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin(nx)\, dx$.

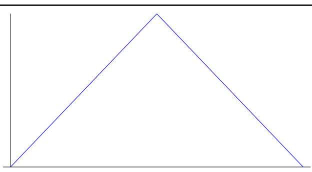
The function is given on $[0,\pi]$.



The odd symmetric extension on $[-\pi,\pi]$.



EXAMPLE. Assume the initial temperature distribution $f(x,0)$ is a sawtooth function which has slope 1 on the interval $[0,\pi/2]$ and slope -1 on the interval $[\pi/2,\pi]$. We first compute the sin-Fourier coefficients of this function.

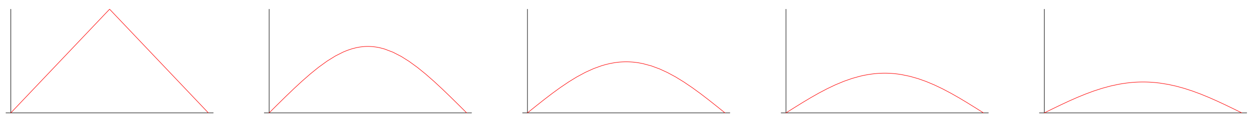


The sin-Fourier coefficients are $b_n = \frac{4}{n^2\pi}(-1)^{(n-1)/2}$ for odd n and 0 for even n. The solution is

$$f(x,t) = \sum_n^{\infty} b_n e^{-\mu n^2 t} \sin(nx) .$$

The exponential term containing the time makes the function $f(x,t)$ converge to 0: The body cools. The higher frequencies are damped faster: "smaller disturbances are smoothed out faster."

VISUALIZATION. We can plot the graph of the function $f(x,t)$ or slice this graph and plot the temperature distribution for different values of the time t.



$f(x,0)$ $f(x,1)$ $f(x,2)$ $f(x,3)$ $f(x,4)$

THE WAVE EQUATION. The height of a string $f(x,t)$ at time t and position x on $[0,\pi]$ satisfies the **wave equation**

$$f_{tt}(t,x) = c^2 f_{xx}(t,x)$$

where c is a constant. As we will see, c is the **speed** of the waves.

REWRITING THE PROBLEM. We can write the problem as

$$\boxed{\frac{d^2}{dt^2} f = c^2 D^2 f}$$

We will solve the problem in the same way as we solved

$$\boxed{\frac{d^2}{dt^2} \vec{x} = A\vec{x}}$$

If A is diagonal, then every basis vector x satisfies an equation of the form $\frac{d^2}{dt^2} x = -c^2 x$ which has the solution $x(t) = x(0)\cos(ct) + x(t)\sin(ct)/c$.

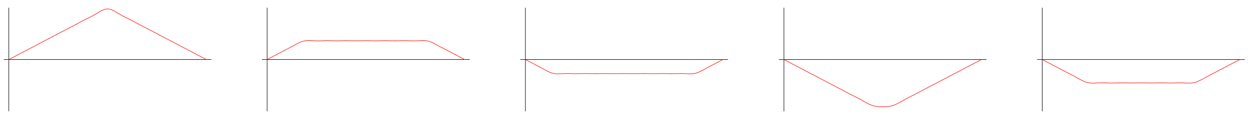
SOLVING THE WAVE EQUATION WITH FOURIER THEORY. The wave equation $f_{tt} = c^2 f_{xx}$ with $f(x,0) = f(x), f_t(x,0) = g(x), f(0,t) = f(\pi,t) = 0$ has the solution

$$f(x,t) = \sum_{n=1}^{\infty} a_n \sin(nx)\cos(nct) + \frac{b_n}{nc}\sin(nx)\sin(nct)$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx$$
$$b_n = \frac{2}{\pi}\int_0^{\pi} g(x)\sin(nx)\,dx$$

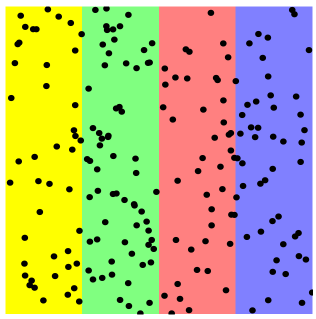
Proof: With $f(x) = \sin(nx), g(x) = 0$, the solution is $f(x,t) = \cos(nct)\sin(nx)$. With $f(x) = 0, g(x) = \sin(nx)$, the solution is $f(x,t) = \frac{1}{c}\sin(ct)\sin(nx)$. For $f(x) = \sum_{n=1}^{\infty} a_n \sin(nx)$ and $g(x) = \sum_{n=1}^{\infty} b_n \sin(nx)$, we get the formula by summing these two solutions.

VISUALIZATION. We can just plot the graph of the function $f(x,t)$ or plot the string for different times t.

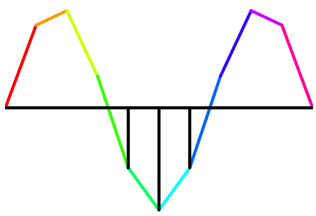


$f(x,0)$ $f(x,1)$ $f(x,2)$ $f(x,3)$ $f(x,4)$

TO THE DERIVATION OF THE HEAT EQUATION. The temperature $f(x,t)$ is proportional to the kinetic energy at x. Divide the stick into n adjacent cells and assume that in each time step, a fraction of the particles moves randomly either to the right or to the left. If $f_i(t)$ is the **energy** of particles in cell i at time t, then the energy of particles at time $t+1$ is proportional to $(f_{i-1}(t) - 2f_i(t) + f_{i+1})(t))$. This is a discrete version of the second derivative because $dx^2 f_{xx}(t,x) \sim (f(x+dx,t) - 2f(x,t) + f(x-dx,t))$.



TO THE DERIVATION OF THE WAVE EQUATION. We can model a string by n discrete particles linked by strings. Assume that the particles can move up and down only. If $f_i(t)$ is the **height** of the particles, then the right particle pulls with a force $f_{i+1} - f_i$, the left particle with a force $f_{i-1} - f_i$. Again, $(f_{i-1}(t) - 2f_i(t) + f_{i+1})(t))$ which is a discrete version of the second derivative because $dx^2 f_{xx}(t,x) \sim (f(x+dx,t) - 2f(x,t) + f(x-dx,t))$.



OVERVIEW: The heat and wave equation can be solved like ordinary differential equations:

Ordinary differential equations	Partial differential equations
$x_t(t) = Ax(t)$ $x_{tt}(t) = Ax(t)$	$f_t(t,x) = f_{xx}(t,x)$ $f_{tt}(t,x) = f_{xx}(t,x)$
Diagonalizing A leads for eigenvectors $\vec{v}$ $Av = -c^2 v$ to the differential equations $v_t = -c^2 v$ $v_{tt} = -c^2 v$ which are solved by $v(t) = e^{-c^2 t} v(0)$ $v(t) = v(0)\cos(ct) + v_t(0)\sin(ct)/c$	Diagonalizing $T = D^2$ with eigenfunctions $f(x) = \sin(nx)$ $Tf = -n^2 f$ leads to the differential equations $f_t(x,t) = -n^2 f(x,t)$ $f_{tt}(x,t) = -n^2 f(x,t)$ which are solved by $f(x,t) = f(x,0)e^{-n^2 t}$ $f(x,t) = f(x,0)\cos(nt) + f_t(x,0)\sin(nt)/n$

NOTATION:

f function on $[-\pi,\pi]$ smooth or piecewise smooth.
t time variable
x space variable
D the partial differential operator $Df(x) = f'(x) = d/dx f(x)$.
T linear transformation, like $Tf = D^2 f = f''$.
c speed of the wave.

$Tf = \lambda f$ Eigenvalue equation analogously to $Av = \lambda v$.
f_t partial derivative of $f(x,t)$ with respect to time t.
f_x partial derivative of $f(x,t)$ with respect to space x.
f_{xx} second partial derivative of f twice with respect to space x.
μ heat conductivity
$f(x) = -f(-x)$ odd function, has sin Fourier series

HOMEWORK

6. Solve the heat equation $f_t = \mu f_{xx}$ on $[0, \pi]$ with the initial condition $f(x, 0) = |\sin(3x)|$ and $f(0, t) = f(\pi, t) = 0$.

THE FOLLOWING THREE EXERCISES (7, 8, 9) BELONG TOGETHER. THEY CONCERN SOLUTIONS TO THE HEAT EQUATION, WHERE THE BOUNDARY VALUES ARE NOT 0.

7. Verify that for any constants a, b the function $h(x, t) = (b - a)\frac{x}{\pi} + a$ is a solution to the heat equation $h_t = \mu h_{xx}$.

8. Assume we have a solution $f(x, t)$ of the heat equation $f_t = \mu f_{xx}$ with $f(0, t) = a$ and $f(\pi, t) = b$. Let $h(x, t)$ be the function from Problem 7. Show that $f(x, t) - h(x, t)$ is a solution of the heat equation $F_t = \mu F_{xx}$ with $F(0, t) = 0$ and $F(\pi, t) = 0$.

9. Solve the heat equation with the initial condition $f(x, 0) = f(x) = \sin(3x) + \frac{x}{\pi}$ and satisfying $f(0, t) = 0$, $f(\pi, t) = 1$ for all times t. This is a situation, when the stick is kept at constant but different temperatures on both ends.

10. A piano string is fixed at the ends $x = 0$ and $x = \pi$ and initially undisturbed. The piano hammer induces an initial velocity $f_t(x, t) = g(x)$ onto the string, where $g(x) = \sin(2x)$ on the interval $\left[0, \frac{\pi}{2}\right]$ and $g(x) = 0$ on $\left[\frac{\pi}{2}, \pi\right]$. Find the motion of the string.