HOMEWORK 1.1: 8,14,20,24,46,26*,36* (Problems with * are recommended only and are not turned in).

SYSTEM OF LINEAR EQUATIONS. A collection of linear equations is called a **system of linear equations**. An example is

$$\left| \begin{array}{ll} 3x - y - z & = 0 \\ -x + 2y - z & = 0 \\ -x - y + 3z & = 9 \end{array} \right| .$$

It consists of three equations for three unknowns x, y, z. **Linear** means that no nonlinear terms like $x^2, x^3, xy, yz^3, \sin(x)$ etc. appear. A formal definition of linearity will be given later.

LINEAR EQUATION. The equation $ax+by = c$ is the general linear equation in two variables and $ax+by+cz = d$ is the general linear equation in three variables. The general **linear equation** in n variables has the form $\boxed{a_1x_1 + a_2x_2 + ... + a_nx_n = a_0}$. Finitely many such equations form a **system of linear equations**.

SOLVING BY ELIMINATION.
Eliminate variables. In the first example, the first equation gives $z = 3x - y$. Putting this into the second and third equation gives

$$\left| \begin{array}{ll} -x + 2y - (3x - y) & = 0 \\ -x - y + 3(3x - y) & = 9 \end{array} \right|$$

or

$$\left| \begin{array}{ll} -4x + 3y & = 0 \\ 8x - 4y & = 9 \end{array} \right| .$$

The first equation gives $y = 4/3x$ and plugging this into the other equation gives $8x - 16/3x = 9$ or $8x = 27$ which means $x = 27/8$. The other values $y = 9/2, z = 45/8$ can now be obtained.

SOLVE BY SUITABLE SUBTRACTION.
Addition of equations. If we subtract the third equation from the second, we get $3y - 4z = -9$ and add three times the second equation to the first, we get $5y - 4z = 0$. Subtracting this equation to the previous one gives $-2y = -9$ or $y = 2/9$.

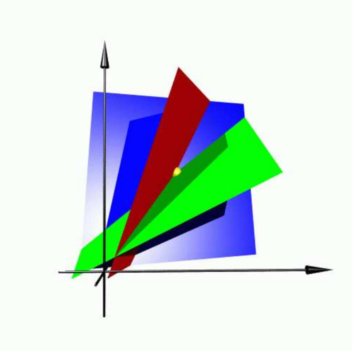
SOLVE BY COMPUTER.
Use the computer. In Mathematica:

$$\text{Solve}[\{3x - y - z == 0, -x + 2y - z == 0, -x - y + 3z == 9\}, \{x, y, z\}] .$$

But what did Mathematica do to solve this equation? We will look in this course at some efficient algorithms.

GEOMETRIC SOLUTION.

Each of the three equations represents a plane in three-dimensional space. Points on the first plane satisfy the first equation. The second plane is the solution set to the second equation. To satisfy the first two equations means to be on the intersection of these two planes which is here a line. To satisfy all three equations, we have to intersect the line with the plane representing the third equation which is a point.



LINES, PLANES, HYPERPLANES.
The set of points in the plane satisfying $ax + by = c$ form a **line**.
The set of points in space satisfying $ax + by + cd = d$ form a **plane**.
The set of points in n-dimensional space satisfying $a_1x_1 + ... + a_nx_n = a_0$ define a set called a **hyperplane**.

RIDDLES:

"15 kids have bicycles or tricycles. Together they count 37 wheels. How many have bicycles?"

Solution. With x bicycles and y tricycles, then $x + y = 15, 2x + 3y = 37$. The solution is $x = 8, y = 7$.

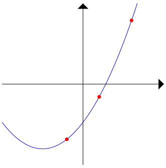
"Tom, the brother of Carry has twice as many sisters as brothers while Carry has equal number of sisters and brothers. How many kids is there in total in this family?"

Solution If there are x brothers and y sisters, then Tom has y sisters and $x-1$ brothers while Carry has x brothers and $y-1$ sisters. We know $y = 2(x-1), x = y-1$ so that $x+1 = 2(x-1)$ and so $x = 3, y = 4$.

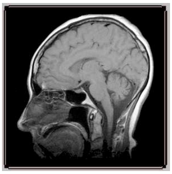
INTERPOLATION.

Find the equation of the parabola which passes through the points $P = (0, -1)$, $Q = (1, 4)$ and $R = (2, 13)$.

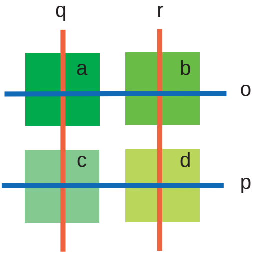
Solution. Assume the parabola is given by the points (x, y) which satisfy the equation $ax^2 + bx + c = y$. So, $c = -1, a+b+c = 4, 4a+2b+c = 13$. Elimination of c gives $a+b = 5, 4a+2b = 14$ so that $2b = 6$ and $b = 3, a = 2$. The parabola has the equation $2x^2+3x-1 = 0$



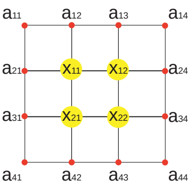
TOMOGRAPHY
Here is a toy example of a problem one has to solve for magnetic resonance imaging (MRI), which makes use of the absorbtion and emission of energy in the radio frequency range of the electromagnetic spectrum.



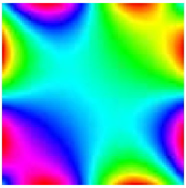
Assume we have 4 hydrogen atoms, whose nuclei are excited with energy intensity a, b, c, d. We measure the spin echo in 4 different directions. $3 = a + b$,$7 = c + d$,$5 = a + c$ and $5 = b + d$. What is a, b, c, d? Solution: $a = 2, b = 1, c = 3, d = 4$. However, also $a = 0, b = 3, c = 5, d = 2$ solves the problem. This system has not a unique solution even so there are 4 equations and 4 unknowns. A good introduction to MRI can be found online at (http://www.cis.rit.edu/htbooks/mri/inside.htm).



INCONSISTENT. $x - y = 4, y + z = 5, x + z = 6$ is a system with no solutions. It is called **inconsistent**.



EQUILIBRIUM. As an example of a system with many variables, consider a drum modeled by a fine net. The heights at each interior node needs the average the heights of the 4 neighboring nodes. The height at the boundary is fixed. With n^2 nodes in the interior, we have to solve a system of n^2 equations. For example, for $n = 2$ (see left), the $n^2 = 4$ equations are $4x_{11} = a_{21}+a_{12}+x_{21}+x_{12}$, $4x_{12} = x_{11}+x_{13}+x_{22}+x_{22}$, $4x_{21} = x_{31}+x_{11}+x_{22}+a_{43}$, $4x_{22} = x_{12}+x_{21}+a_{43}+a_{34}$. To the right, we see the solution to a problem with $n = 300$, where the computer had to solve a system with $90'000$ variables.



LINEAR OR NONLINEAR?
a) **The ideal gas law** $PV = nKT$ for the P, V, T, the pressure p, volume V and temperature T of a gas.
b) **The Hook law** $F = k(x - a)$ relates the force F pulling a string extended to length x.
c) **Einsteins mass-energy equation** $E = mc^2$ relates restmass m with the energy E of a body.

ON THE HISTORY. In 2000 BC the Babylonians already studied problems which led to linear equations. In 200 BC, the Chinese used a method similar to Gaussian elimination to solve systems of linear equations.