MATRIX PRODUCT. If B is a $n \times p$ matrix and A is a $p \times m$ matrix, then BA is defined as the $n \times m$ matrix with entries $(BA)_{ij} = \sum_{k=1}^{m} B_{ik} A_{kj}$.

EXAMPLE. If B is a 3×4 matrix, and A is a 4×2 matrix then BA is a 3×2 matrix.

$$B = \left[\begin{array}{cccc} 1 & 3 & 5 & 7 \\ 3 & 1 & 8 & 1 \\ 1 & 0 & 9 & 2 \end{array} \right], A = \left[\begin{array}{cc} 1 & 3 \\ 3 & 1 \\ 1 & 0 \\ 0 & 1 \end{array} \right], BA = \left[\begin{array}{cccc} 1 & 3 & 5 & 7 \\ 3 & 1 & 8 & 1 \\ 1 & 0 & 9 & 2 \end{array} \right] \left[\begin{array}{cc} 1 & 3 \\ 3 & 1 \\ 1 & 0 \\ 0 & 1 \end{array} \right] = \left[\begin{array}{cc} 15 & 13 \\ 14 & 11 \\ 10 & 5 \end{array} \right].$$

COMPOSING LINEAR TRANSFORMATIONS. If $S : \mathbf{R}^m \to \mathbf{R}^p, x \mapsto Ax$ and $T : \mathbf{R}^p \to \mathbf{R}^n, y \mapsto By$ are linear transformations, then their composition $T \circ S : x \mapsto B(A(x))$ is a linear transformation from $\mathbf{R}^m$ to $\mathbf{R}^n$. The corresponding $n \times m$ matrix is the matrix product BA.

EXAMPLE. Find the matrix which is a composition of a rotation around the x-axes by an agle $\pi/2$ followed by a rotation around the z-axes by an angle $\pi/2$.
SOLUTION. The first transformation has the property that $e_1 \to e_1, e_2 \to e_3, e_3 \to -e_2$, the second $e_1 \to e_2, e_2 \to -e_1, e_3 \to e_3$. If A is the matrix belonging to the first transformation and B the second, then BA is the matrix to the composition. The composition maps $e_1 \to -e_2 \to e_3 \to e_1$ is a rotation around a long diagonal. $B = \left[\begin{array}{ccc} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{array} \right] A = \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{array} \right], BA = \left[\begin{array}{ccc} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{array} \right].$

EXAMPLE. A rotation dilation is the composition of a rotation by $\alpha = \arctan(b/a)$ and a dilation (=scale) by $r = \sqrt{a^2 + b^2}$.

REMARK. Matrix multiplication can be seen a generalization of usual multiplication of numbers and also generalizes the dot product.

MATRIX ALGEBRA. Note that $AB \neq BA$ in general and A^{-1} does not always exist, otherwise, the same rules apply as for numbers:
$A(BC) = (AB)C$, $AA^{-1} = A^{-1}A = 1_n$, $(AB)^{-1} = B^{-1}A^{-1}$, $A(B + C) = AB + AC$, $(B + C)A = BA + CA$ etc.

PARTITIONED MATRICES. The entries of matrices can themselves be matrices. If B is a $n \times p$ matrix and A is a $p \times m$ matrix, and assume the entries are $k \times k$ matrices, then BA is a $n \times m$ matrix, where each entry $(BA)_{ij} = \sum_{l=1}^{p} B_{il} A_{lj}$ is a $k \times k$ matrix. Partitioning matrices can be useful to improve the speed of matrix multiplication

EXAMPLE. If $A = \left[\begin{array}{cc} A_{11} & A_{12} \\ 0 & A_{22} \end{array} \right]$, where A_{ij} are $k \times k$ matrices with the property that A_{11} and A_{22} are invertible, then $B = \left[\begin{array}{cc} A_{11}^{-1} & -A_{11}^{-1}A_{12}A_{22}^{-1} \\ 0 & A_{22}^{-1} \end{array} \right]$ is the inverse of A.
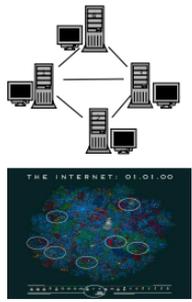

NETWORKS. Let us associate to the computer network a matrix $\left[\begin{array}{cccc} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{array} \right]$ A worm in the first computer is associated to $\left[\begin{array}{c} 1 \\ 0 \\ 0 \\ 0 \end{array} \right]$. The vector Ax has a 1 at the places, where the worm could be in the next step. The vector $(AA)(x)$ tells, in how many ways the worm can go from the first computer to other hosts in 2 steps. In our case, it can go in three different ways back to the computer itself.
Matrices help to solve combinatorial problems (see movie "Good will hunting"). For example, what does $[A^{1000}]_{22}$ tell about the worm infection of the network? What does it mean if A^{100} has no zero entries?

FRACTALS. Closely related to linear maps are **affine maps** $x \mapsto Ax + b$. They are compositions of a linear map with a translation. It is **not** a linear map if $B(0) \neq 0$. Affine maps can be disguised as linear maps in the following way: let $y = \begin{bmatrix} x \\ 1 \end{bmatrix}$ and define the $(n+1)*(n+1)$ matrix $B = \begin{bmatrix} A & b \\ 0 & 1 \end{bmatrix}$. Then $By = \begin{bmatrix} Ax + b \\ 1 \end{bmatrix}$.

Fractals can be constructed by taking for example 3 affine maps R, S, T which contract space. For a given object Y_0 define $Y_1 = R(Y_0) \cup S(Y_0) \cup T(Y_0)$ and recursively $Y_k = R(Y_{k-1}) \cup S(Y_{k-1}) \cup T(Y_{k-1})$. The above picture shows Y_k after some iterations. In the limit, for example if $R(Y_0), S(Y_0)$ and $T(Y_0)$ are disjoint, the sets Y_k converge to a **fractal**, an object with dimension strictly between 1 and 2.
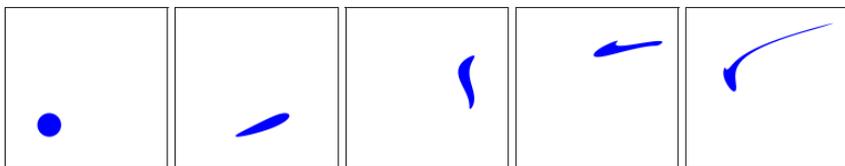
CHAOS. Consider a map in the plane like $T : \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} 2x + 2\sin(x) - y \\ x \end{bmatrix}$. We apply this map again and again and follow the points $(x_1, y_1) = T(x, y)$, $(x_2, y_2) = T(T(x, y))$, etc. Lets write T^n for the n-th iteration of the map and (x_n, y_n) for the image of (x, y) under the map T^n. The linear approximation of the map at a point (x, y) is the matrix $DT(x, y) = \begin{bmatrix} 2 + 2\cos(x) - 1 \\ 1 \end{bmatrix}$. (If $T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} f(x, y) \\ g(x, y) \end{bmatrix}$, then the row vectors of $DT(x, y)$ are just the gradients of f and g). T is called **chaotic at** (x, y), if the entries of $D(T^n)(x, y)$ grow exponentially fast with n. By the **chain rule**, $D(T^n)$ is the product of matrices $DT(x_i, y_i)$. For example, T is chaotic at $(0, 0)$. If there is a positive probability to hit a chaotic point, then T is called chaotic.

FALSE COLORS. Any color can be represented as a vector (r, g, b), where $r \in [0, 1]$ is the red $g \in [0, 1]$ is the green and $b \in [0, 1]$ is the blue component. Changing colors in a picture means applying a transformation on the cube. Let $T : (r, g, b) \mapsto (g, b, r)$ and $S : (r, g, b) \mapsto (r, g, 0)$. What is the composition of these two linear maps?

OPTICS. Matrices help to calculate the motion of light rays through lenses. A light ray $y(s) = x + ms$ in the plane is described by a vector (x, m). Following the light ray over a distance of length L corresponds to the map $(x, m) \mapsto (x + mL, m)$. In the lens, the ray is bent depending on the height x. The transformation in the lens is $(x, m) \mapsto (x, m - kx)$, where k is the strength of the lense.

$$\begin{bmatrix} x \\ m \end{bmatrix} \mapsto A_L \begin{bmatrix} x \\ m \end{bmatrix} = \begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ m \end{bmatrix}, \quad \begin{bmatrix} x \\ m \end{bmatrix} \mapsto B_k \begin{bmatrix} x \\ m \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -k & 1 \end{bmatrix} \begin{bmatrix} x \\ m \end{bmatrix}.$$

Examples:
1) Eye looking far: $A_R B_k$. 2) Eye looking at distance L: $A_R B_k A_L$.
3) Telescope: $B_{k_2} A_L B_{k_1}$. (More about it in problem 80 in section 2.4).