Name:

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- Start by writing your name in the above box and check your section in the box to the left.
- Try to answer each question on the same page as the question is asked. If needed, use the back or the next empty page for work. If you need additional paper, write your name on it.
- Do not detach pages from this exam packet or unstaple the packet.
- Please write neatly and except for problems 1-3, give details. Answers which are illegible for the grader can not be given credit.
- No notes, books, calculators, computers, or other electronic aids can be allowed.
- You have 90 minutes time to complete your work.

1		20
2		10
3		10
4		10
5		10
6		10
7		10
8		10
9		10
10		10
Total:		110

Problem 1) TF questions (20 points) No justifications are needed.

1) | T | **F** | A rotation in the plane around the point $(1,1)$ by angle 90^0 is a linear transformation.

Solution:
The transformation does not leave $\vec{0} = (0,0)$ invariant.

2) | **T** | F | If $ABC = I_2$ for 2×2 matrices A, B, C, then A is invertible.

Solution:
If A were not invertible, then the image of A were not the entire space, but then also the image of ABC were not the entire space and ABC were not invertible.

3) | T | **F** | There is a 2×3 matrix A and a 3×2 matrix B such that $AB = BA$.

Solution:
The matrix AB is a 2×2 matrix and the matrix BA is a 3×3 matrix.

4) | T | **F** | For any linear system $Ax = b$ with 3×3 matrix A, the augmented 3×4 matrix $B = [A|b]$ satisfies rank(A) = rank(B).

Solution:
No, the rank of B is larger than the rank of A if the system has no solution.

5) | T | **F** | If the system $Ax = b$ has a unique solution for some b, then A must be a square matrix.

Solution:
No, it is possible that the matrix has more rows than columns and still that we have a unique solution.

6) | **T** | F | If $v_1, \dots, v_4$ are linearly independent vectors in $\mathbf{R}^4$, then they must form a basis for $\mathbf{R}^4$.

Solution:
We have to show that they also span. Look at the matrix A which contains $v_1, ..., v_4$ as columns. This matrix has no kernel. By the rank-nullety theorem, they there image is the entire space and the vectors also span.

7) | T | F | For every 3×3 matrix A, we have that $\text{rref}(A)$ is similar to A.

Solution:
Take the matrix $A = 3I_3$. It satisfies $\text{rref}(A) = I_3$. The only matrix similar to I_3 is I_3.

8) | T | F | For any two 3×3 matrices A, B the identity $(A+B)(A+B) = A^2+2AB+B^2$ holds.

Solution:
This is wrong if $AB \neq BA$.

9) | T | F | The set X of quadratic polynomials satisfying $f'(0) = f(0)$ is a linear space.

Solution:
Check the three properties $0 \in X, f, g \in X$ then $f + g \in X$ and $\lambda f \in X$.

10) | T | F | If A is a non-invertible matrix then $\text{rref}(A)$ has at least one row of zero.

Solution:
Let A be the identity matrix I_2. Delete the last row. Then we have a noninvertible 1×2 matrix which has no rows with zeros. It is noninvertible already because it is not a square matrix.

11) | T | F | The plane $2x + 3y + 5z - 10 = 0$ is the image of a linear transformation T.

Solution:
The image of a linear transformation is a linear space. Especially, it has to contain the origin. The plane under consideration does not contain 0.

12) T F — For any $n \times n$ matrix A the identity $\ker(A^2) = \ker(\text{rref}(A^2))$ holds.

Solution:
The kernel does not change under row reduction.

13) T F — For any $n \times n$ matrix A, the intersection of the kernel $\ker(A)$ and the image $\text{im}(A)$ is the trivial space $\{0\}$.

Solution:
The matrix $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ has the same kernel and image.

14) T F — There is a 2×2 matrix for which $A^6 = -I_2$.

Solution:
Look for rotations. $-I_2$ is a rotation by 180 degrees. So, a rotation by 360/12=30 degrees will do.

15) T F — A 3×3 matrix A can satisfy $\ker(A) = \text{im}(A)$.

Solution:
This is not possible due to the rank-nullety theorem.

16) T F — If the linear system $A\vec{x} = \vec{b}$ has exactly one solution for some vector $\vec{b}$, then it has exactly one solution for all vectors $\vec{b}$.

Solution:
Having exactly one solution means that the kernel of A is trivial. But it can happen that we have a solution for one b and no solution for an other b. An example is $Ax = b$ with $A = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $b = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ but no solution for $b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. The vector b has to be in the image of A.

17) T F — If $\mathcal{B} = \{\vec{v}_1, \vec{v}_2\}$ is a basis for $\mathbb{R}^2$, then $\vec{v}_1 \cdot \vec{v}_2 = 0$.

Solution:
There is a basis where the vectors are not orthogonal.

18) | T | F | The rank of an $m \times n$ matrix is at most m.

Solution:
There are maximally m and maximally n leading 1 in the matrix.

19) | T | F | If A is a 4×3 matrix and rref(A) has exactly two nonzero rows, then $\dim(\ker(A)) = 1$.

Solution:
If there are two nonzero rows, then we have two leading 1 and so the dimension of the image is 2. By the dimension formula, the dimension of the kernel is 1.

20) | T | F | A reflection in space about a line is similar to a rotation by 90° around the z axis.

Solution:
A reflection satisfies $A^2 = I_n$ and a rotation by 90 degrees satisfies B^2 does not satisfy $B^2 = I_2$.

| | Total

Problem 2) (10 points) No justifications are needed.

Match each of the following matrices with the geometric descriptions.

Matrix	Enter A-G here.
a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$	
b) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	
c) $\begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$	
d) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	
e) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	

A) Projection onto a plane.
B) Rotation around an axis.
C) Reflection at a point.
D) Projection onto a line.
E) Reflection at a plane.
F) Reflection at a line.
G) Identity transformation.

Solution:
The correct entries are: F,A,B,D,E

Problem 3) (10 points)

a) (6 points) Check the box to the left if the set is a linear space. Here P_n denotes the space of all polynomials of degree $\leq n$ and $C[0,1]$ denotes the linear space of all continuous functions on the interval $[0,1]$.

	$X = \{$ All 3×3 matrices A for which all row vectors v have the same length $\|v\| = 1$ $\}$
	$X = \{f$ in $C[0,1] \mid \int_0^1 f(x)\,dx = 0\}$
	$X = \{f$ in $P_4 \mid f(1) + f(4) = 0\}$
	$X = \{f$ in $P_4 \mid f(1)\cdot f(4) = 1\}$
	$X = \{$ All 2×2 matrices A for which the second row contains only zeros $\}$
	$X = \{$ All 2×2 matrices A for which at least one column contains only zeros. $\}$

b) (4 points) In the basis $\mathcal{B} = \{\begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\}$, the matrix $A = \begin{bmatrix} 0 & 2 \\ 0 & -2 \end{bmatrix}$ becomes one of the following two matrices. Check the correct one.

	$B = \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix}$
	$B = \begin{bmatrix} 0 & 4 \\ 0 & -2 \end{bmatrix}$

Solution:
a) No, yes, yes, no, yes, no.
b) We have $S^{-1}AS = \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix}$ and $SAS^{-1} = \begin{bmatrix} 0 & 4 \\ 0 & -2 \end{bmatrix}$. The first matrix is the right one.

Problem 4) (10 points)

Consider the system of linear equations

$$\begin{aligned} x + y + z + w &= a \\ 2x + 2y + 2z + 2w &= b \\ 3x + 3y + 3z + 3w &= c\,. \end{aligned}$$

a) (4 points) Are there a, b, c such that the system has exactly one solution? If yes, find such a, b, c. If not, why not?

b) (3 points) Are there a, b, c such that the system has no solution? If yes, find such a, b, c. If not, why not?

c) (3 points) Are there a, b, c such that the system has infinitely many solutions? If yes, find such a, b, c. If not, why not?

Solution:
a) $\boxed{\text{No}}$. The matrix A of the system $Ax = b$ has a nontrivial kernel. There are ∞ or 0 solutions.
b) $\boxed{\text{Yes}}$. Take $a = 1, b = 1, c = 1$ for example.
c) $\boxed{\text{yes}}$ For $a = 1, b = 2, c = 3$ we have infinitely many solutions. All equations are in this case equivalent to one equation.

Problem 5) (10 points)

Let A be the matrix which is the reflection about the x-axis in the plane. Let B be the matrix of the transformation which leaves e_1 the same and maps e_2 to $e_1 + e_2$.
Finally, define $C = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$.

a) (6 points) Find the product $AB^{100}C$.

b) (4 points) What is $(ABC)^{10}$?

Solution:

a) The three matrices are $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

$C = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$

We have $B^{100} = \begin{bmatrix} 1 & 100 \\ 0 & 1 \end{bmatrix}$

The product is

$$D = AB^{100}C = \begin{bmatrix} 0 & 200 \\ 0 & -2 \end{bmatrix}$$

b) To see what happens when we take powers of $D = ABC$ form $D^2, D^3, ...$ and watch for a pattern:

$$D = ABC = \begin{bmatrix} 0 & -2 \\ 0 & -2 \end{bmatrix} D^2 = \begin{bmatrix} 0 & -4 \\ 0 & 4 \end{bmatrix}, D^3 = \begin{bmatrix} 0 & 8 \\ 0 & -8 \end{bmatrix}, D^4 = \begin{bmatrix} 0 & -16 \\ 0 & 16 \end{bmatrix}$$

etc. We see that the lower diagonal entry always changes sign and the last column always stays parallel and doubles its length: therefore

$$D^{10} = \begin{bmatrix} 0 & -1024 \\ 0 & 1024 \end{bmatrix} .$$

Problem 6) (10 points)

The matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 1 & 2 \\ 4 & 3 & 2 & 1 \end{bmatrix}$$

is called a **latin square**. The sum of the entries of each row and column vectors add up to the same number 10.

Part of **Melancolia I**, an engraving by the German Renaissance master Albrecht Dürer. This master piece was done in 1514. It contains a latin square which is even a **magic square**: The diagonals add up to the same number too and all matrix elements are different. Additionally, the date 1514 appears in the bottom row of the matrix. By the way: the student in **Melancolia I** takes a linear algebra exam - therefore the name of the picture.

a) (4 points) In the matrix A, the sum of the first and last column is equal to the sum of the second and third column. Find a nonzero vector in the kernel of A.

b) (6 points) Find a basis for the image of A.

Solution:

a) Because the relation between the columns is given, we know that $\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$ is in the kernel.

b) Row reduction shows that the first three rows span the image.

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} .$$

The basis is

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 4 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 1 \\ 2 \end{bmatrix} \right\} .$$

Problem 7) (10 points)

Consider the vectors

$$\vec{a} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \vec{b} = \begin{bmatrix} 3 \\ 9 \\ 2 \end{bmatrix}, \vec{c} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}, \vec{d} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$$

in $\mathbf{R^3}$. Let V be the set of all vectors $\vec{x}$ in $\mathbf{R^3}$ for which all the dot products

$$\vec{a} \cdot \vec{x},\ \vec{b} \cdot \vec{x},\ \vec{c} \cdot \vec{x},\ \vec{d} \cdot \vec{x}$$

are zero.

a) (4 points) Find a 4×3 matrix A whose kernel is V.

b) (4 points) Find a basis of V

c) (2 points) What is the dimension of V?

Solution:
a) We write the vectors as rows a matrix A and find the kernel of A:

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 3 & 9 & 2 \\ 0 & 0 & 2 \\ 1 & 3 & 1 \end{bmatrix} .$$

Matrix multiplication has the property that if $Ax = 0$, then x is perpendicular to all row vectors.
b) To find the kernel, we row reduce the matrix A:

$$\text{rref}(A) = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} .$$

The kernel is spanned by $\begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$. It has the basis $\mathcal{B} = \{ \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} \}$.
c) The dimension is the number of elements in the basis of the kernel which is $\boxed{1}$.

Problem 8) (10 points)

Let A be a matrix for which

$$B = \text{rref}(A) = \begin{bmatrix} 1 & 3 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} .$$

a) (6 points) Find a basis for the kernel of A and determine the dimension of the kernel and image of A.

b) (4 points) Is it possible that A was a matrix for which $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{bmatrix}$ is in the image of A? If yes, give a matrix A. If no, argue why not.

Solution:

a) The kernel of A is the kernel of B. It is an important property of row reduction that the kernel does not change with row operations. It is spanned by $\begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}$. By the dimension formula, the dimension of the image is 3 and the dimension of the kernel is 2.

b) $\boxed{\text{Yes}}$, we can add multiples of the first row to all other rows. An explicit example can be obtained by adding 2 times the first row to the second row, 3 times the first row to the third row etc.

$$A = \begin{bmatrix} 1 & 3 & 0 & 1 & 0 \\ 2 & 6 & 1 & 4 & 0 \\ 3 & 9 & 0 & 3 & 1 \\ 4 & 12 & 0 & 4 & 0 \\ 5 & 15 & 0 & 5 & 0 \\ 6 & 18 & 0 & 6 & 0 \end{bmatrix}$$

Problem 9) (10 points)

Let

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} .$$

a) (5 points) Find the inverse of A.

b) (5 points) A 3×3 matrix X satisfies the equation

$$AXA = A - I_3$$

Find X.

Solution:
a) We can get this by row reducing the 3×6 matrix $[A|I_3]$:

$$A^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} .$$

b) Multiply the equation from the left and from the right with A^{-1}:

$$X = A^{-1} - A^{-2} = \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} .$$

Problem 10) (10 points)

Let A be a 3×4 and let B be a 4×3 matrix.

a) (3 points) Can the 3×3 matrix AB be invertible?

b) (3 points) Can the 4×4 matrix BA be invertible?

c) (4 points) Find a 3×4 matrix A and a 4×3 matrix B so that AB and BA both have the same rank 2.

Solution:
a) Yes, it can be the identity matrix. If B has basis vectors as columns and A has basis vectors as rows. Here is an example

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} .$$

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} .$$

$AB = I_3$.
b) No, the rank of BA must be smaller or equal than 3: Because the rank of A can be maximally 3, and there are 4 columns of A, there must be a one dimensional kernel by the rank-nullelty theorem. Let v be a nonzero vector in that kernel. Then $Av = 0$ and so $BAv = 0$. This shows that the matrix BA has a kernel which is at least one dimensional. But by the rank-nullelty theorem again, BA has a maximally 3 dimensional rank. A 4×4 matrix of rank 3 or less can not be invertible.
c) Yes it is possible:

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} .$$

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} .$$

Then $BA = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and $AB = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.