

Math21b Fall 2010 Review Second Hourly

December 31, 2010

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Content:

- Orthogonal matrices
- Projection
- Least square solutions, Data fitting
- Determinants
- Eigenvalues and Eigenvectors
- Similarity and Diagonalization
- Complex Numbers

orthogonal matrix

$A =$

$$\begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

$/2$

Column Vectors form
Orthonormal basis

$$A^T A = I$$

1. Orthogonal Matrices

Compare

$A =$

1	1	-1
1	1	1
1	-1	1
1	-1	-1

not
orthogonal
basis.

$/2$

AA^T

projection

Column Vectors are
orthonormal basis of image

$$A^T A = I_3$$

Orthogonal Transformation

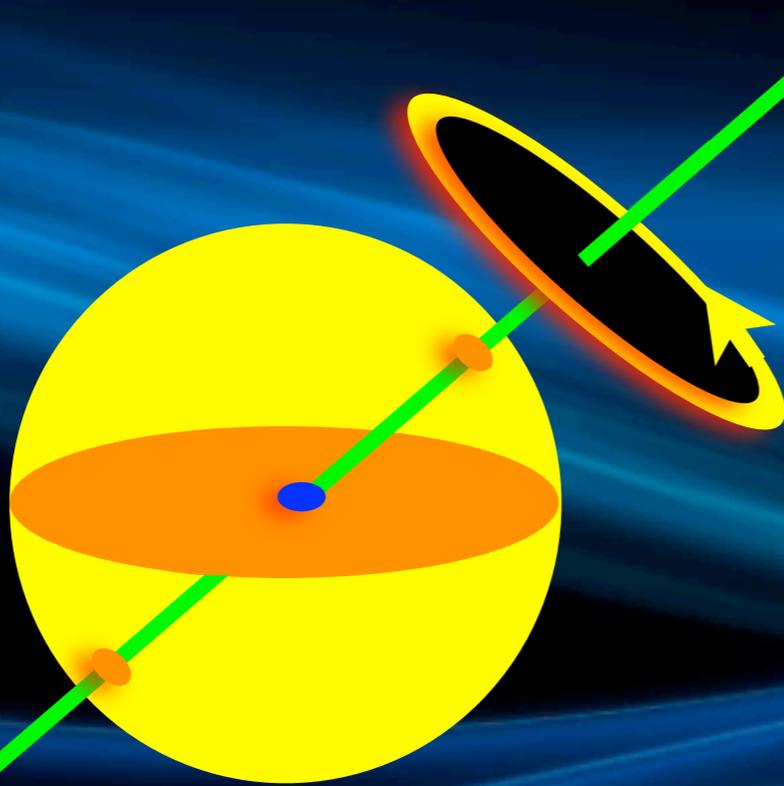
$$\mathbf{x} \rightarrow A(\mathbf{x}) \quad A^T A = I_n$$

- Preserves length and angles
- Determinant is 1 or -1
- Combination of reflections and rotations

$$\det(Q)=1$$

example:

eigenvector



$$Q =$$

$\cos(t)$	0	$-\sin(t)$
0	1	0
$\sin(t)$	0	$\cos(t)$

Rotations



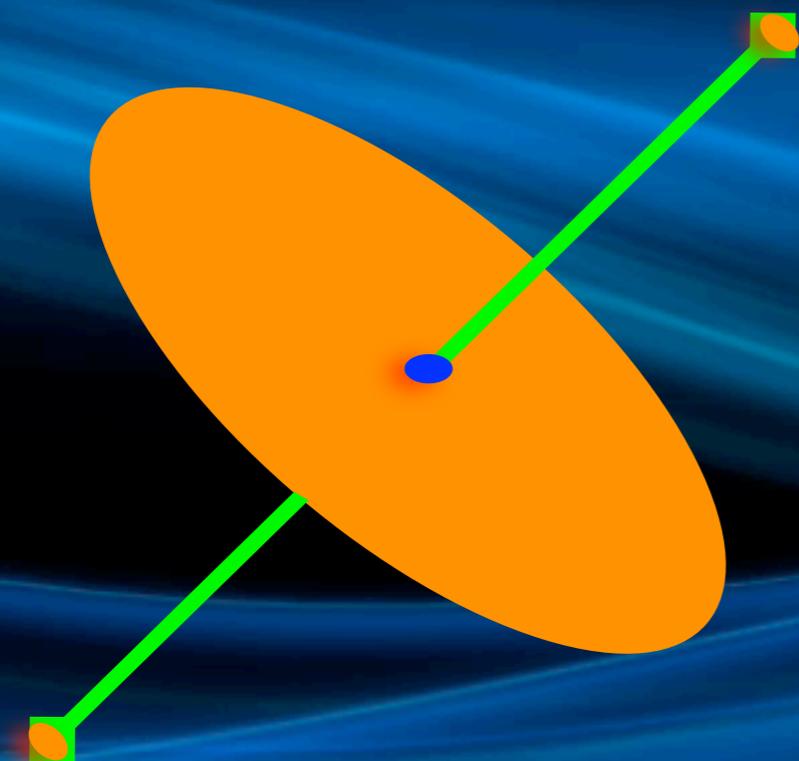
Reflections

$$\det(Q) = -1$$

if reflected at odd dimensional space

$$\det(Q) = +1$$

if reflected at even dimensional space



$Q =$

$\cos(t)$	$\sin(t)$	0
$\sin(t)$	$-\cos(t)$	0
0	0	



1. Projections

Orthogonal Projections

$\det(A) = 0$
except for
identity

If orthonormal basis in V is known:

$$A = Q Q^T$$

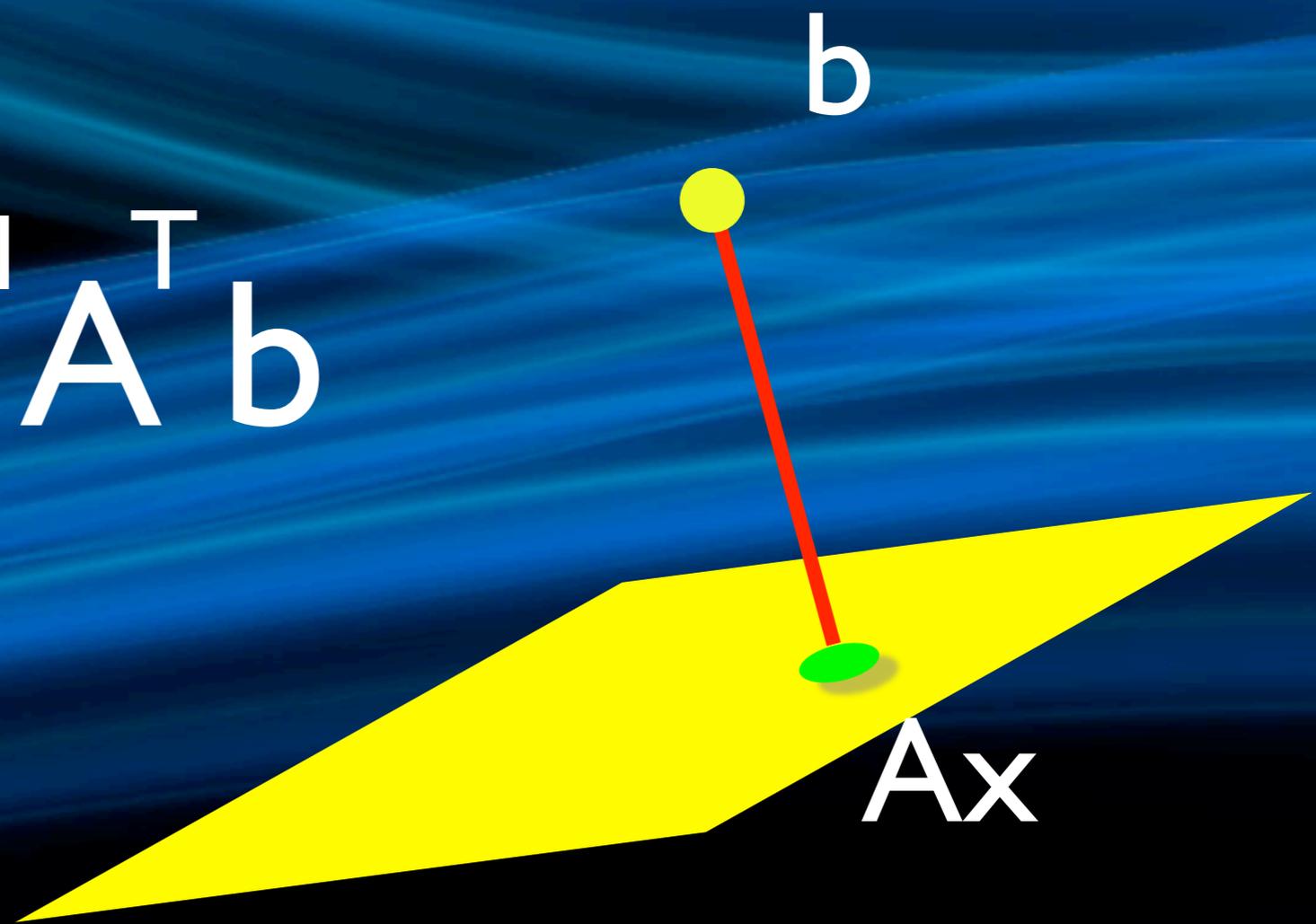
$$A^2 = Q Q^T Q Q^T = A$$

Least Square Solutions

$$Ax=b$$

$$x = (A^T A)^{-1} A^T b$$

$$A^T (Ax-b) = 0$$



Projection

$$P y = A(A^T A)^{-1} A^T y$$

simplifies to

$$P y = A A^T y$$

if columns of A are orthonormal.

Problem:

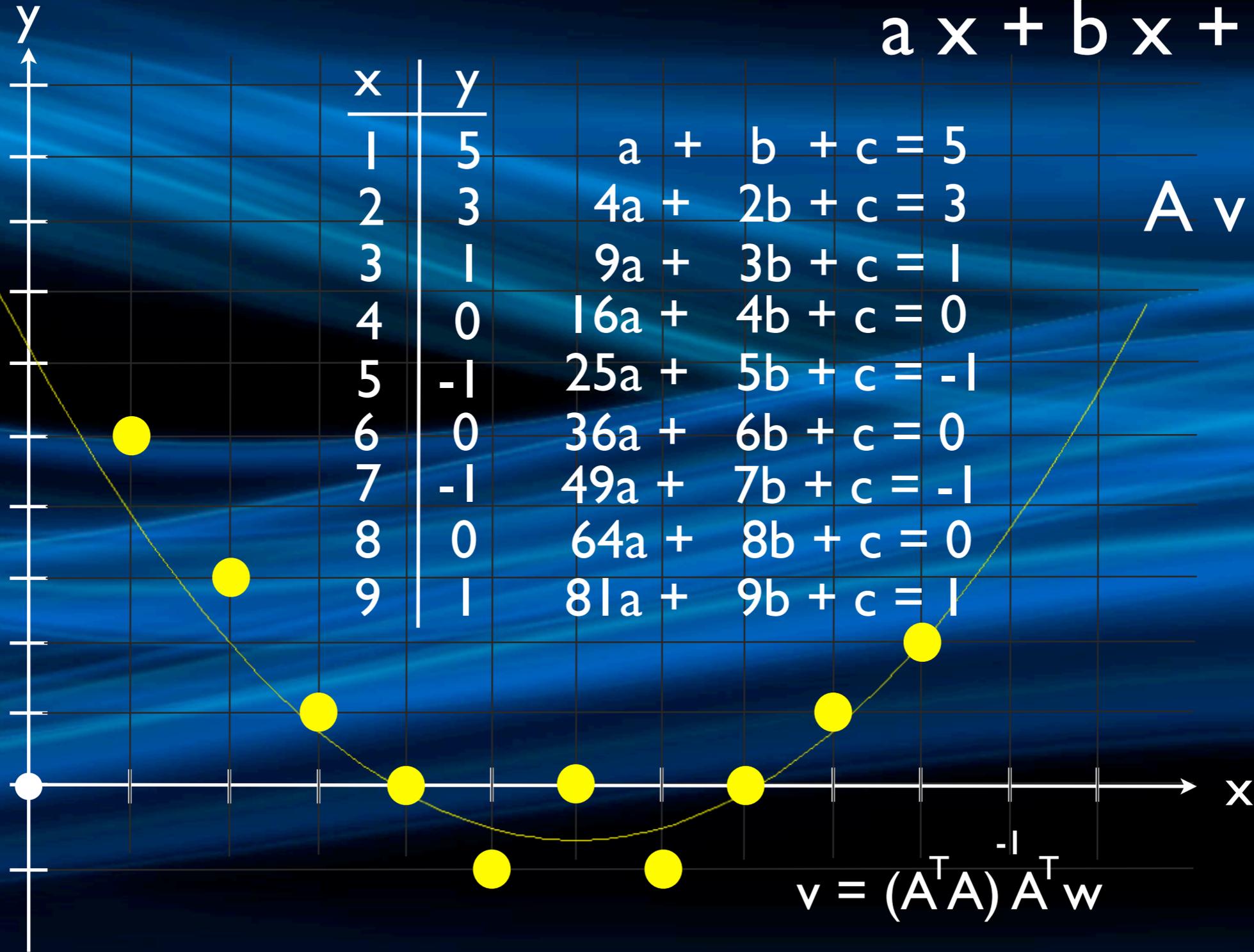
Find the projection on the space with basis:

$$\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \\ 1 \end{bmatrix} \right\} \quad P = \frac{\begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 3 & 0 & 3 & 0 \\ 2 & 0 & 4 & 0 & 2 \\ 0 & 3 & 0 & 3 & 0 \\ 1 & 0 & 2 & 0 & 1 \end{bmatrix}}{6}$$

Data Fitting:

write down equations
as if all points are

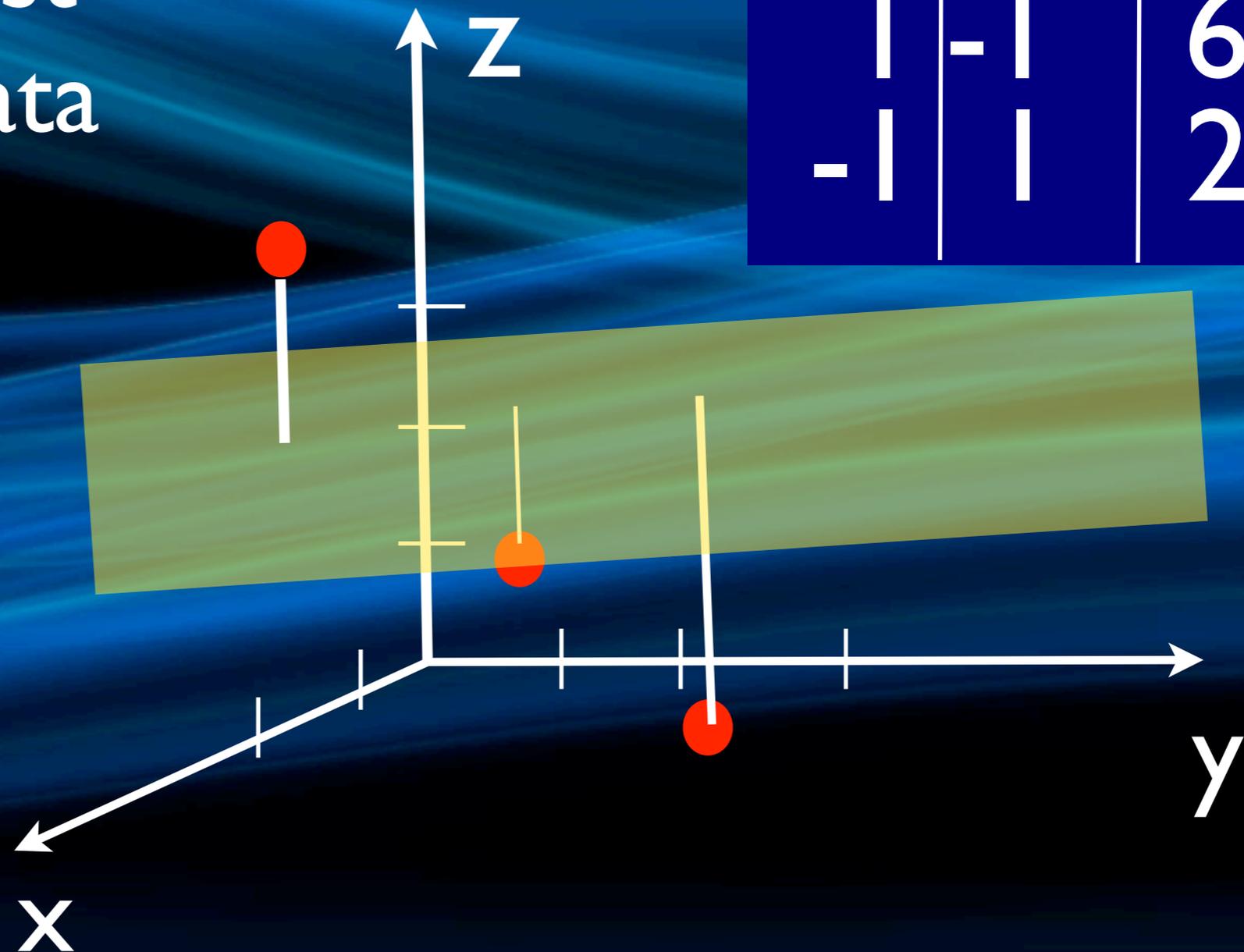
$$ax^2 + bx + c = y$$



Problem:

Find the plane $z=ax+by$
which best
fits the data

x	y	z
-2	3	0
1	-1	6
-1	1	2



Solution:

x	y	z
-2	3	0
1	-1	6
-1	1	2

$$A^T =$$

3	-1	1
-2	1	-1

$$A =$$

-2	3
1	-1
-1	1

$$A^T b =$$

4
-4

$$(A^T A)^{-1} A^T b =$$

6
4

$$A^T A =$$

6	-8
-8	11

$$(A^T A)^{-1} =$$

11	8
8	6

/2

The Projection

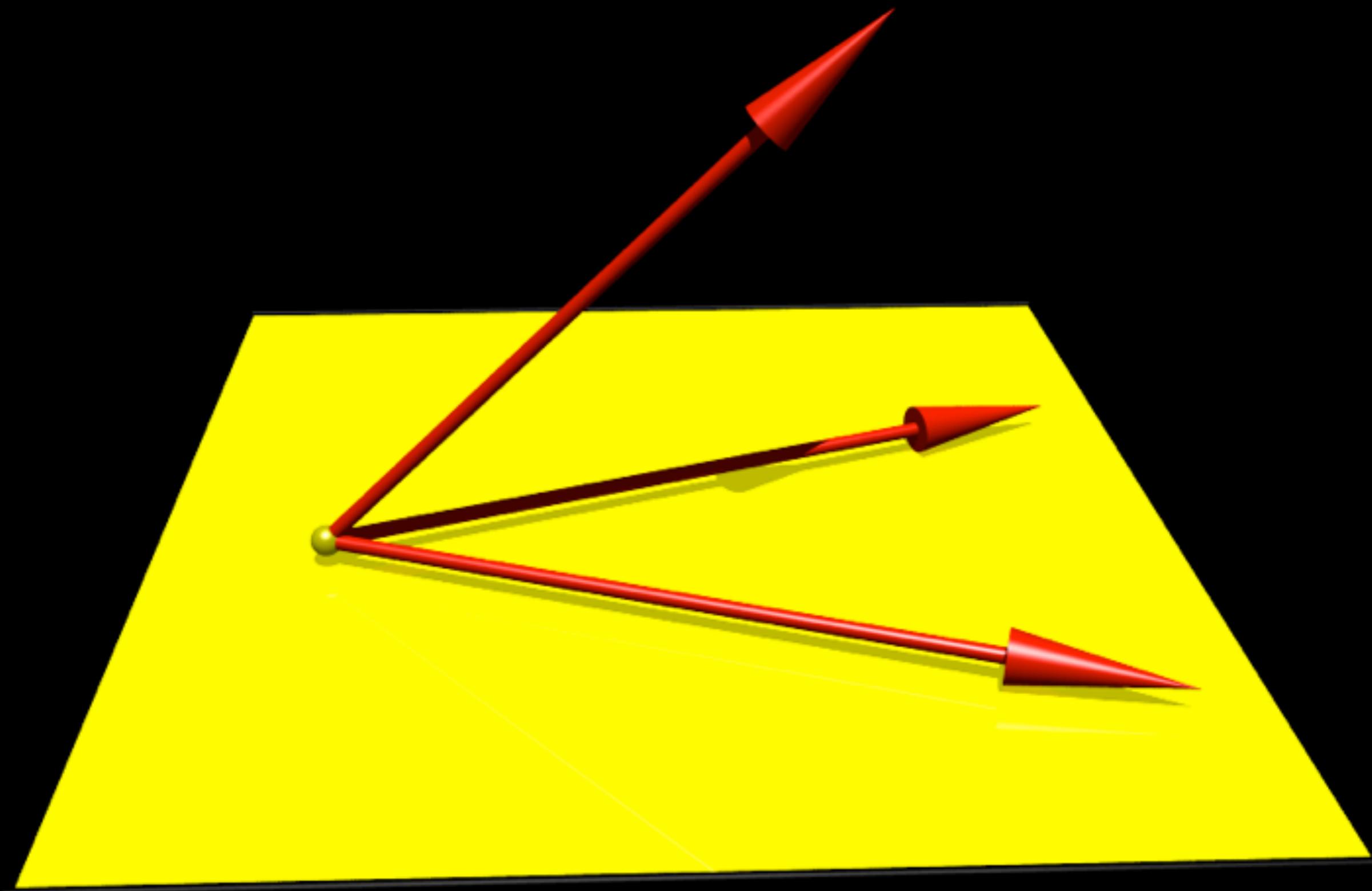
$$P = A (A^T A)^{-1} A^T$$

$A^T A$ invertible

$$P = A A^T \quad \text{if } A \text{ has orthonormal columns}$$

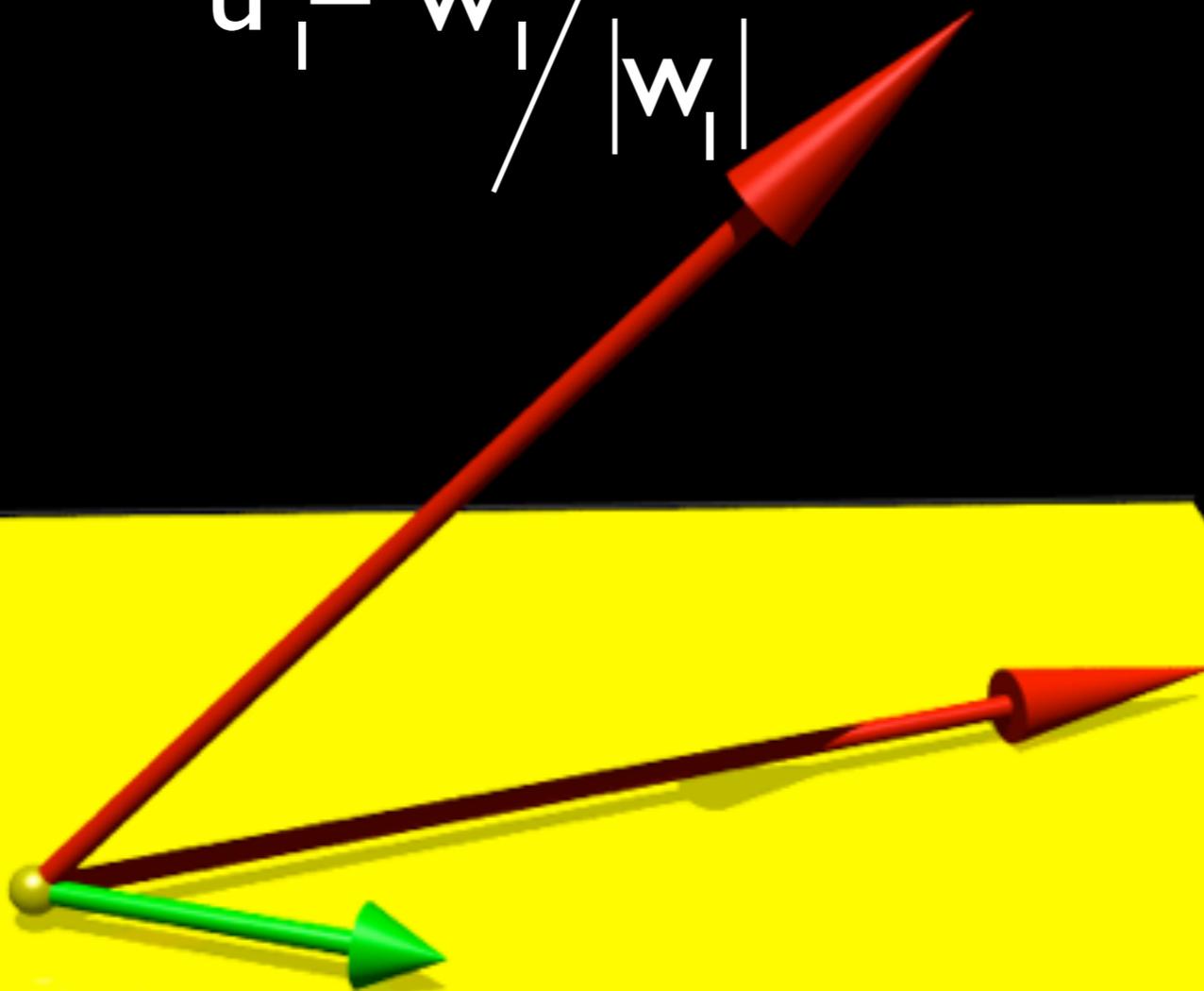
3. Gram-Schmidt

Straighten out and Scale



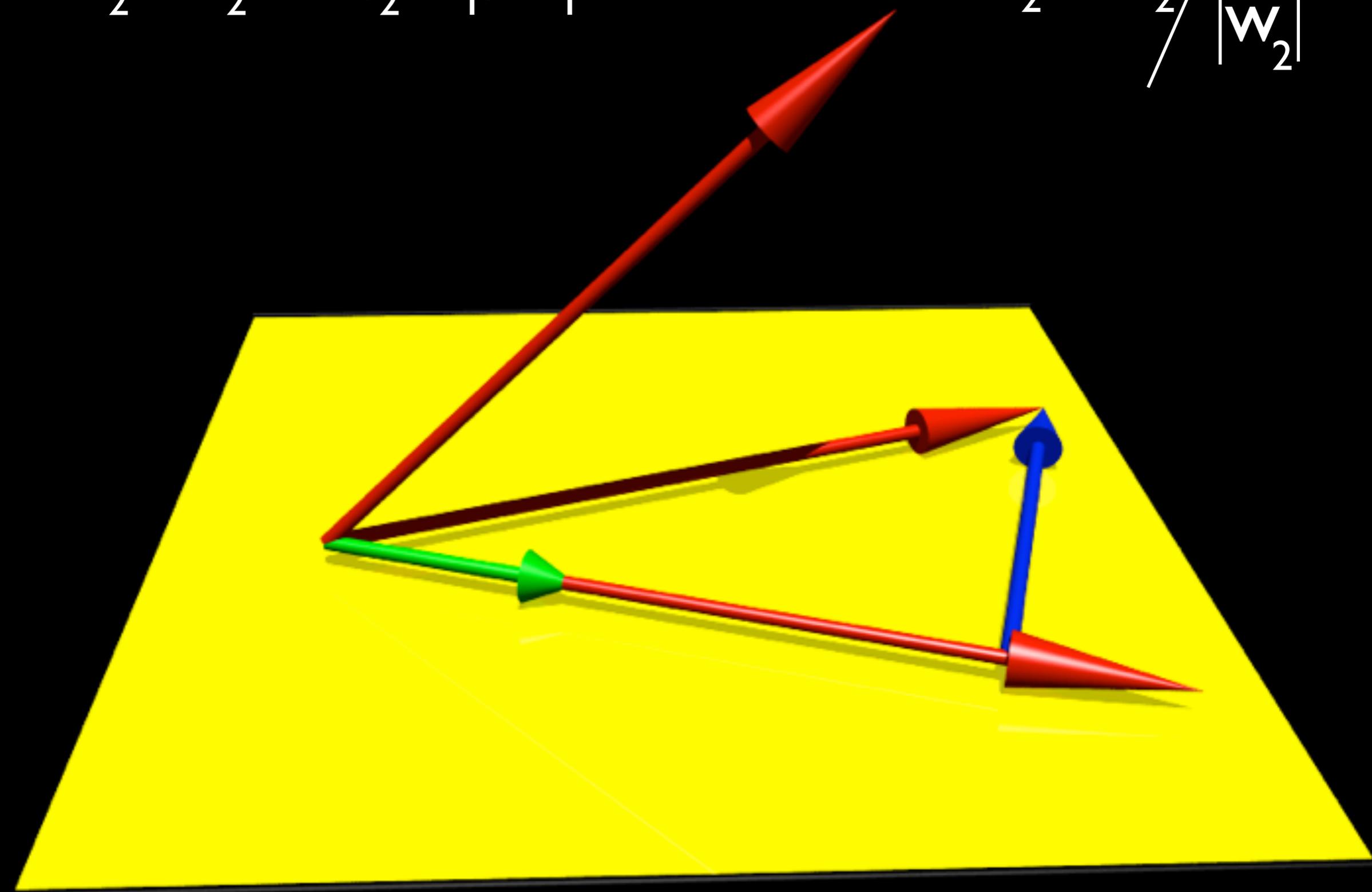
$$w_1 = v_1$$

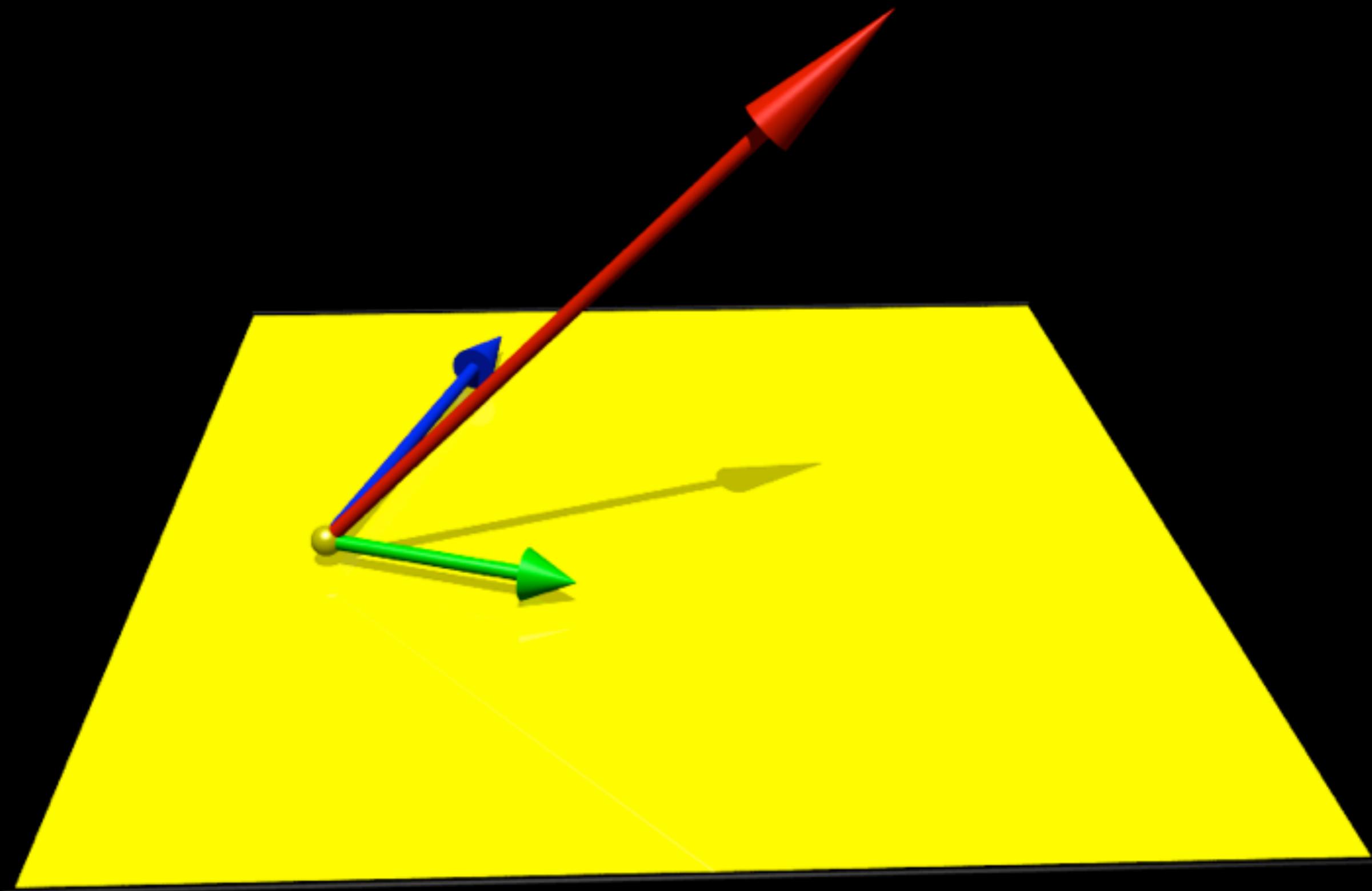
$$u_1 = w_1 / |w_1|$$



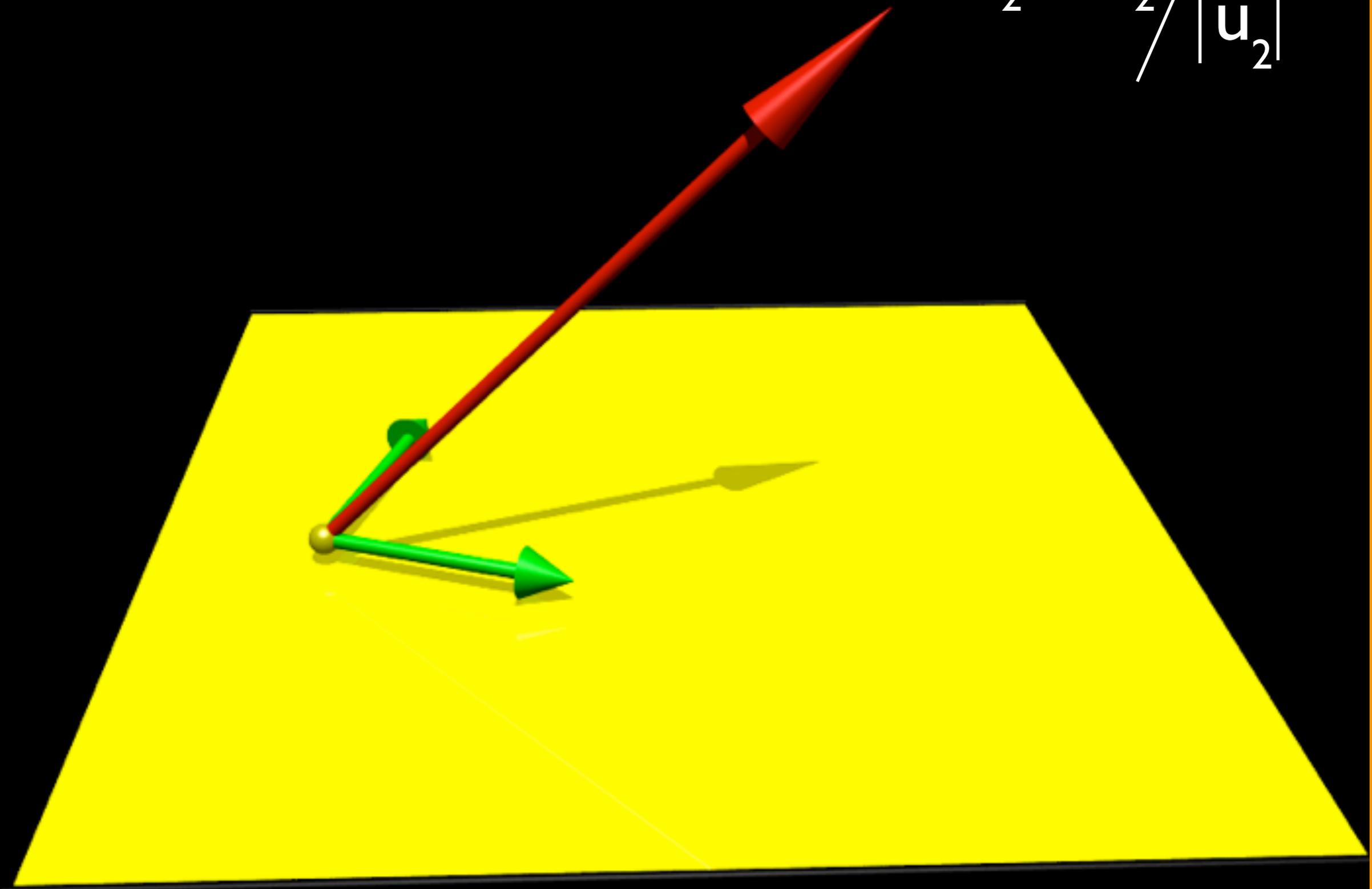
$$\mathbf{w}_2 = \mathbf{v}_2 - (\mathbf{v}_2 \cdot \mathbf{u}_1) \mathbf{u}_1$$

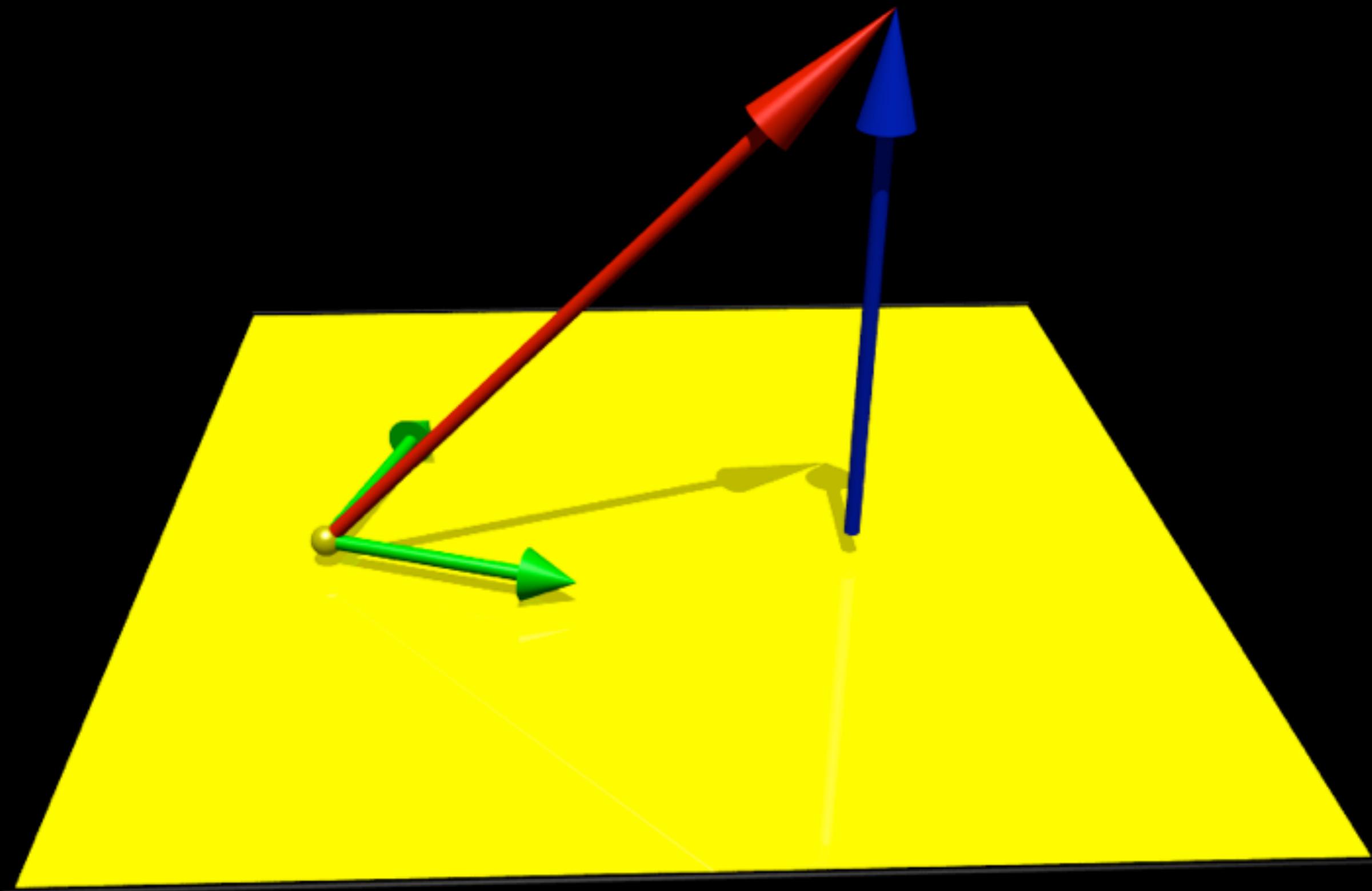
$$\mathbf{u}_2 = \mathbf{w}_2 / |\mathbf{w}_2|$$





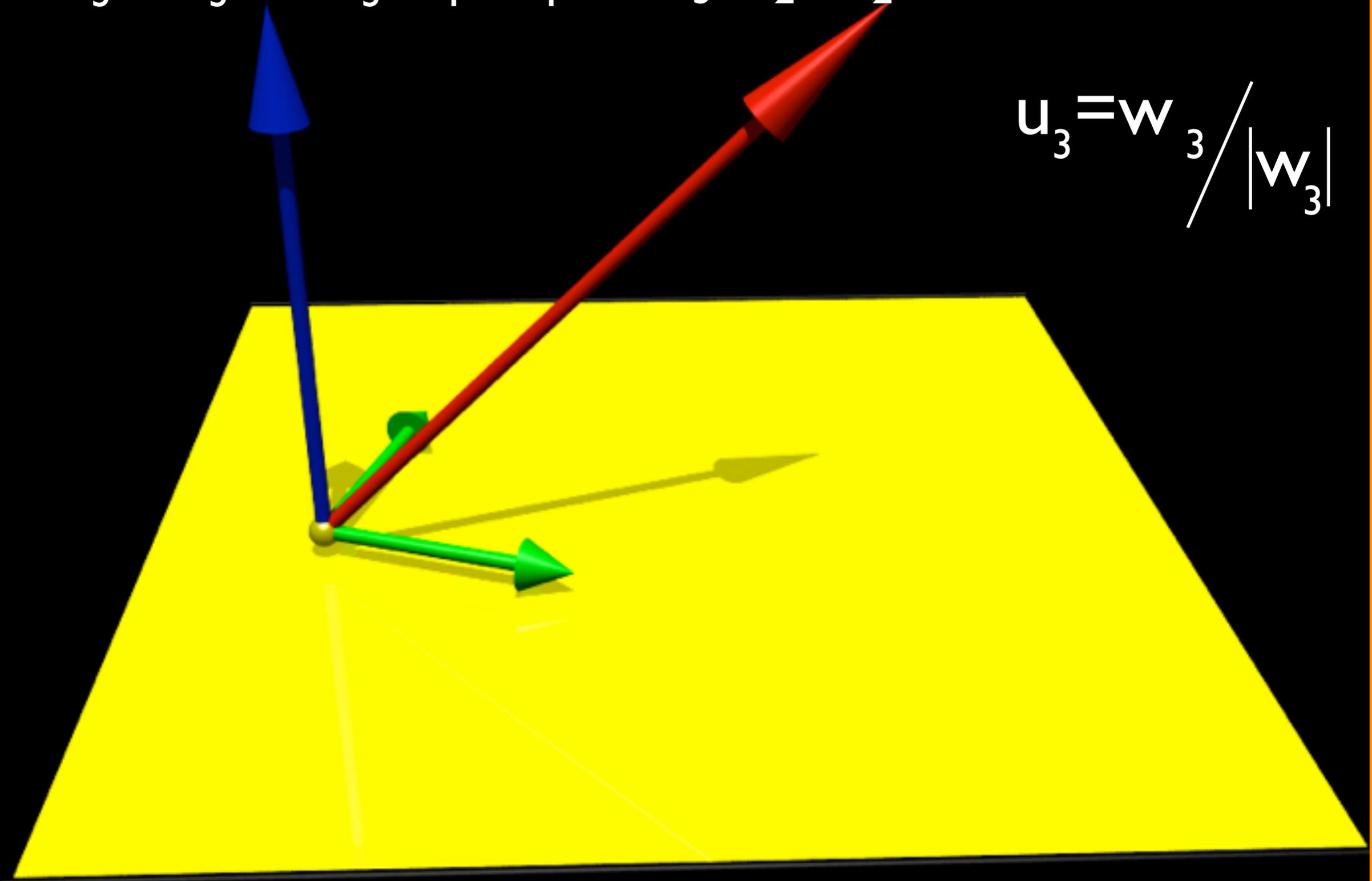
$$w_2 = u_2 / |u_2|$$

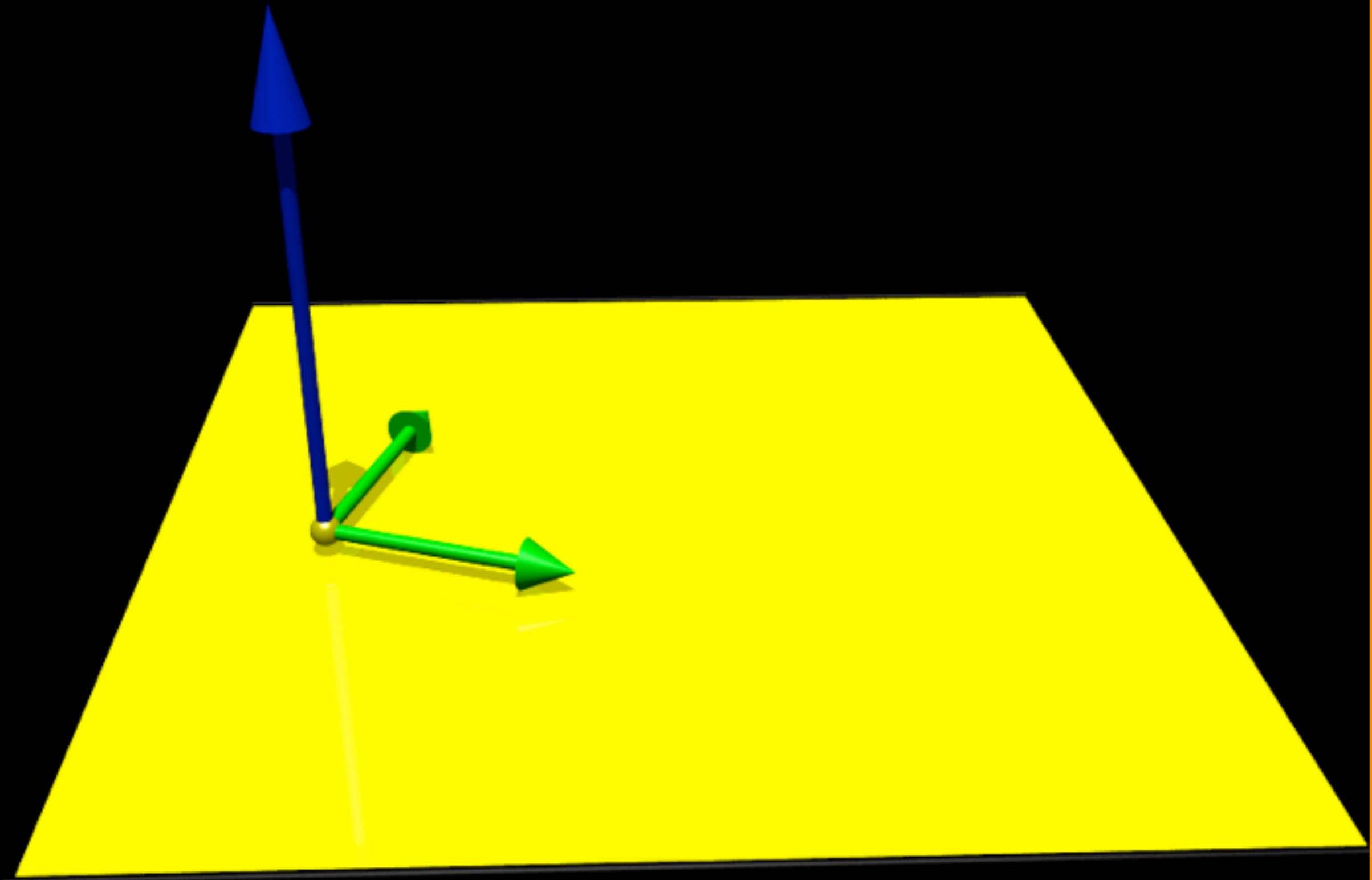


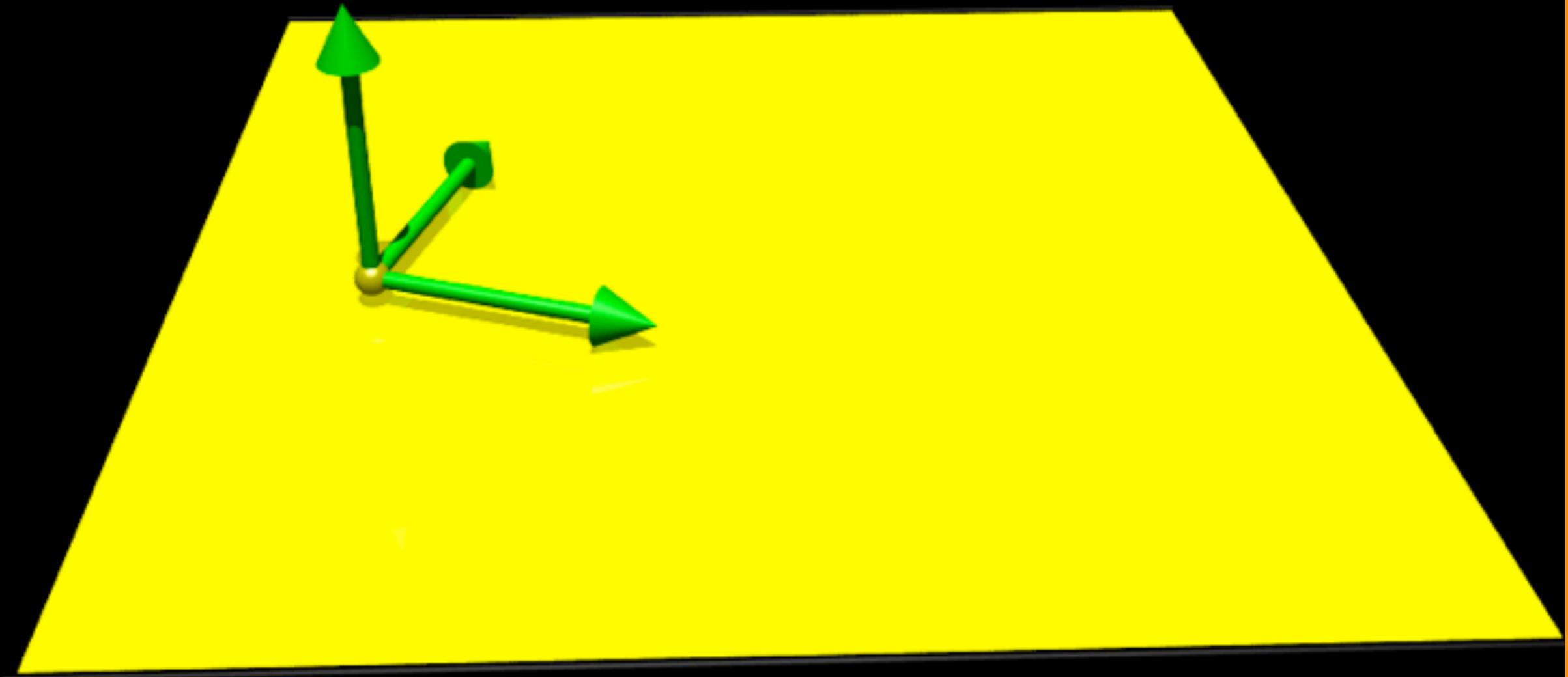


$$\mathbf{w}_3 = \mathbf{v}_3 - (\mathbf{v}_3 \cdot \mathbf{u}_1) \mathbf{u}_1 - (\mathbf{v}_3 \cdot \mathbf{u}_2) \mathbf{u}_2$$

$$\mathbf{u}_3 = \mathbf{w}_3 / |\mathbf{w}_3|$$



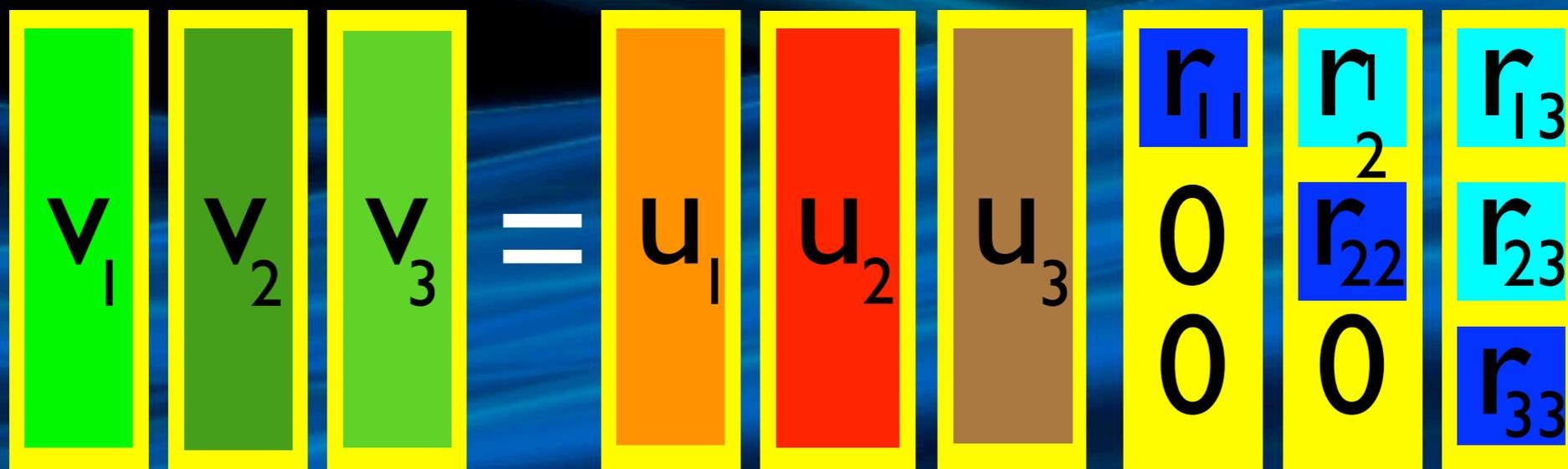




QR-decomposition

$$A = QR$$

$$r_{ii} = |w_i| \quad r_{ij} = (w_i \cdot v_j)$$



Problem:

perform Gram-Schmidt

$$\left\{ \begin{array}{c} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{array}, \begin{array}{c} 1 \\ 1 \\ 2 \\ 1 \\ 1 \end{array} \right\}$$

$Q =$

$$\begin{array}{cc} 0 & 1/\sqrt{6} \\ 1/\sqrt{2} & 0 \\ 0 & 2/\sqrt{6} \\ 1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{6} \end{array}$$

Problem:

a):

2	3	5
0	6	4
0	0	7

Find the QR factorization

b):

0	0	-2
0	2	0
2	0	0

One and two dimensions

$$\det(\begin{bmatrix} a \end{bmatrix}) = a$$

$$\det(\begin{bmatrix} a & b \\ c & d \end{bmatrix}) = ad - bc$$

$$\det \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} =$$

$$+ \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} - \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

$$aei + bfg + cdh - bdi - ceg - ahf$$

Anagram Memo

$$\det\left(\begin{array}{|c|c|c|} \hline a & e & s \\ \hline o & i & u \\ \hline d & t & m \\ \hline \end{array}\right) =$$

mia + deu + sot

-tua - dis -emo

Anagram Memo

det(

a	e	s
o	i	u
d	t	m

) =

mia + deu + sot
-tua - dis -emo



Still play it



Laplace expansion

\det

a			
b			
c			
d			

=

a		

\det

$-$ **b**

$+$ **c**

$-$ **d**

\det

Partitioned Matrices

$$\det\left(\begin{array}{cc} \boxed{A} & \boxed{C} \\ \boxed{0} & \boxed{B} \end{array}\right) =$$

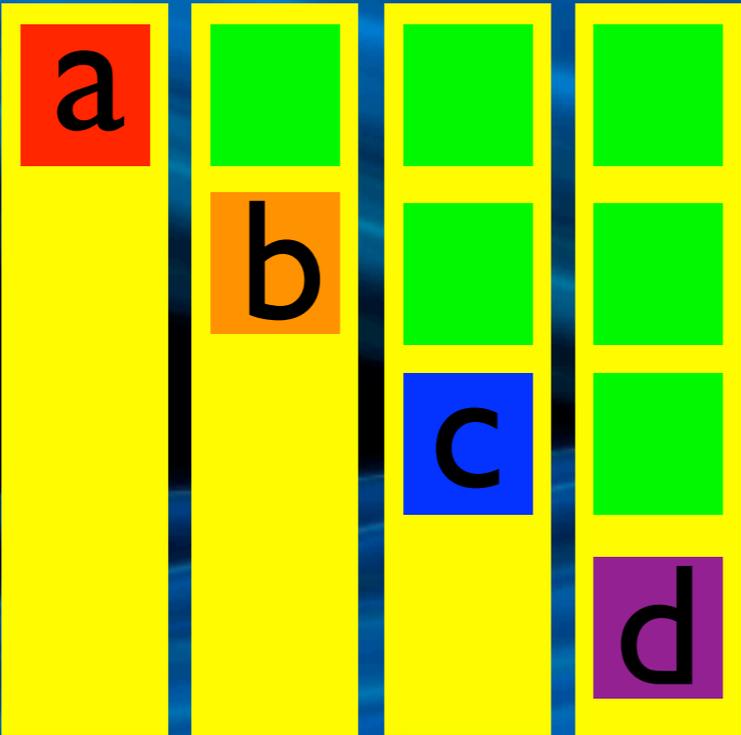
$$\det(\boxed{A}) \det(\boxed{B})$$

Similarly

$$\det\left(\begin{array}{|c|c|} \hline \mathbf{A} & \mathbf{0} \\ \hline \mathbf{C} & \mathbf{B} \\ \hline \end{array}\right) = \det(\mathbf{A}) \det(\mathbf{B})$$

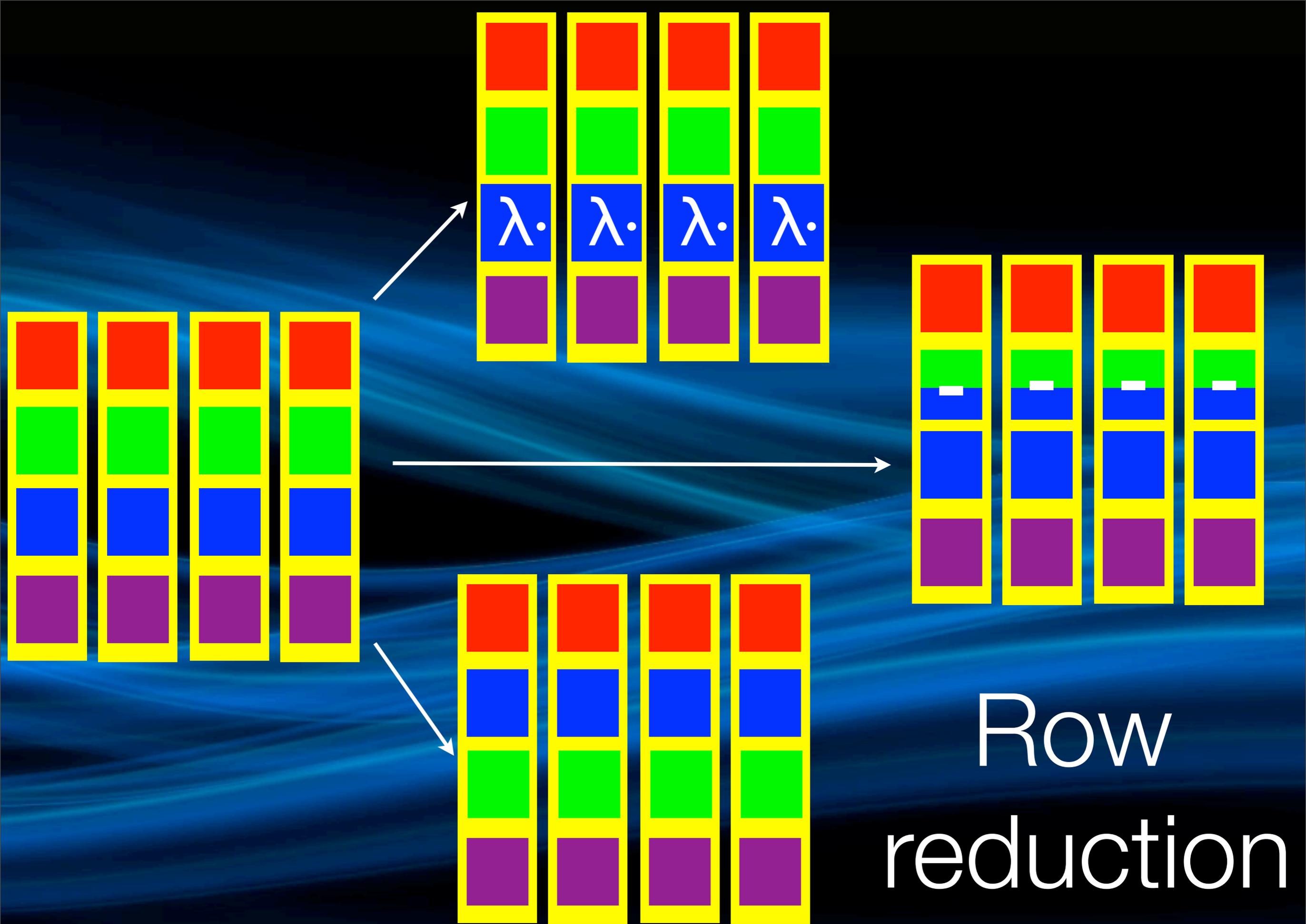
Triangular

det



= a b c d

The diagram illustrates the determinant of a triangular matrix. On the left, the word "det" is written in white. To its right is a 4x4 matrix with four columns highlighted in yellow. The diagonal elements are highlighted in colored boxes: 'a' in red, 'b' in orange, 'c' in blue, and 'd' in purple. The upper triangular elements are highlighted in green. An equals sign follows, leading to the product of the diagonal elements: 'a b c d', where each letter is in a colored box matching its position in the diagonal.



Row
reduction

Problem: find

det

2	5	3	4	5	2
1	2	3	3	4	3
0	0	2	0	0	0
0	0	3	3	0	0
0	0	4	2	2	0
0	0	5	1	2	1

DVD to win

Murder

DVD to win



shining

DVD Problem:

0	0	0	0	0	0	0	2	0
5	5	5	0	0	0	3	2	2
0	0	0	0	0	0	2	3	0
2	1	0	0	0	0	0	0	0
3	2	-2	0	0	0	0	0	0
4	3	3	2	0	0	0	0	0
4	3	3	3	-1	0	0	5	0
3	2	2	2	2	2	0	5	0
1	0	0	0	0	0	0	5	0

5. Eigenvalues

$$A v = \lambda v$$

$$f(\lambda) = \det(A - \lambda I)$$

$$v \in \ker(A - \lambda I)$$

eigenspace

Examples:

- Orthogonal: 1, -1, or complex
- Projections: 1, 0
- Reflections: 1, -1
- Shear: 1
- Regular Transition matrix: 1

2x2 Matrices

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$f(\lambda) = \lambda^2 - \text{tr}(A)\lambda + \det(A)$$

$$\lambda_1 = \frac{\text{tr}(A) + \sqrt{\text{tr}(A)^2 - 4\det(A)}}{2}$$

$$\lambda_2 = \frac{\text{tr}(A) - \sqrt{\text{tr}(A)^2 - 4\det(A)}}{2}$$

Algebraic-Geometric multiplicities

algebraic

number of simultaneous
roots of λ .

geometric

dimension of $\ker(A-\lambda I)$

a	b
0	a

$$f_A(\lambda) = (\lambda - a)(\lambda - a)$$

algebraic multiplicity: 2
geometric multiplicity: 1

Formulas

$$\det(A) = \lambda_1 \lambda_2 \dots \lambda_n$$

$$\operatorname{tr}(A) = \lambda_1 + \lambda_2 + \dots + \lambda_n$$

$$\det(A^k) = \lambda_1^k \lambda_2^k \dots \lambda_n^k$$

$$\operatorname{tr}(A^k) = \lambda_1^k + \lambda_2^k + \dots + \lambda_n^k$$

Problem:

Find the $\text{tr}(A)$ for a noninvertible 2×2 matrix for which

$$\det(A + 5I_2) = 10$$

See the determinant?

1	1	1	2	0
1	3	1	0	0
0	0	0	0	4



An other determinant

det

1	1	1	1	1	9
9	1	1	1	1	1
1	9	1	1	1	1
1	1	9	1	1	1
1	1	1	9	1	1
1	1	1	1	9	1



$$A = \begin{bmatrix} 1 & 2 \\ 20 & 3 \end{bmatrix}$$

23

-9

$$A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

$a+ib$

$a-ib$

$$A = \begin{bmatrix} 1/2 & 1/5 \\ 1/2 & 4/5 \end{bmatrix}$$

1

-3/10

Find eigenvectors:

$$A = \begin{bmatrix} 2 & -3 & 0 & 0 \\ 3 & 2 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 7 & 5 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

6. Discrete Dynamical systems

$$x(t+1) = A x(t)$$

$$x(0) = c_1 v_1 + \dots + c_n v_n$$

$$A v_k = \lambda_k v_k$$

$$x(t) = c_1 \lambda_1^t v_1 + \dots + c_n \lambda_n^t v_n$$

1, 1, 2, 3, 5, 8, 13, 21, ...



$$x(n+1) = x(n) + x(n-1) \quad 1, 1, 2, 3, 5, 8, 13, 21, \dots$$

$$\begin{bmatrix} x(n+1) \\ x(n) \end{bmatrix} = \begin{bmatrix} x(n) + x(n-1) \\ x(n) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x(n) \\ x(n-1) \end{bmatrix}$$

characteristic polynomial: $\lambda^2 - \lambda - 1 = f_A(\lambda)$

eigenvalues = $\varphi, 1-\varphi$

eigenvectors =

$$\begin{bmatrix} \varphi \\ 1 \end{bmatrix} \quad \begin{bmatrix} 1-\varphi \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \varphi \\ 1 \end{bmatrix} - \begin{bmatrix} 1-\varphi \\ 1 \end{bmatrix}$$

$$x(n) = \frac{\varphi^n - (1-\varphi)^n}{\sqrt{5}}$$

Olivers sun flower, 2005

$$\varphi = (1 + \sqrt{5})/2$$

golden ratio φ



$$\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \begin{array}{l} \text{new branches} \\ \text{old branches} \end{array}$$

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix}$$



$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$



$$\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$



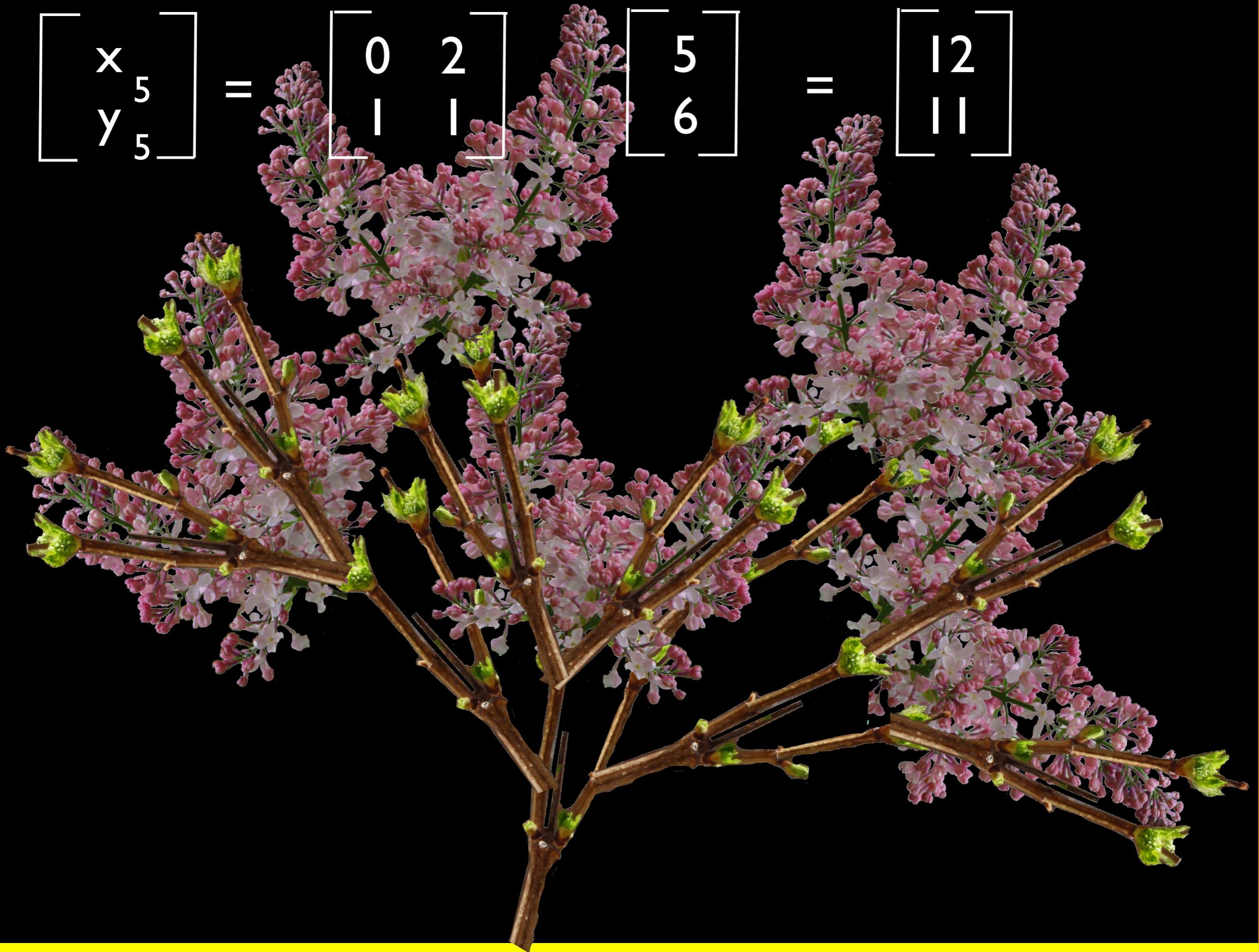
$$\begin{bmatrix} x_3 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$



$$\begin{bmatrix} x_4 \\ y_4 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \end{bmatrix}$$



$$\begin{bmatrix} x_5 \\ y_5 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 12 \\ 11 \end{bmatrix}$$



7) Similarity and Diagonalization

Necessary for Similarity:

	Trace the same
	Same determinant
	Same eigenvalues
	Powers are similar
	Same rank and nullity

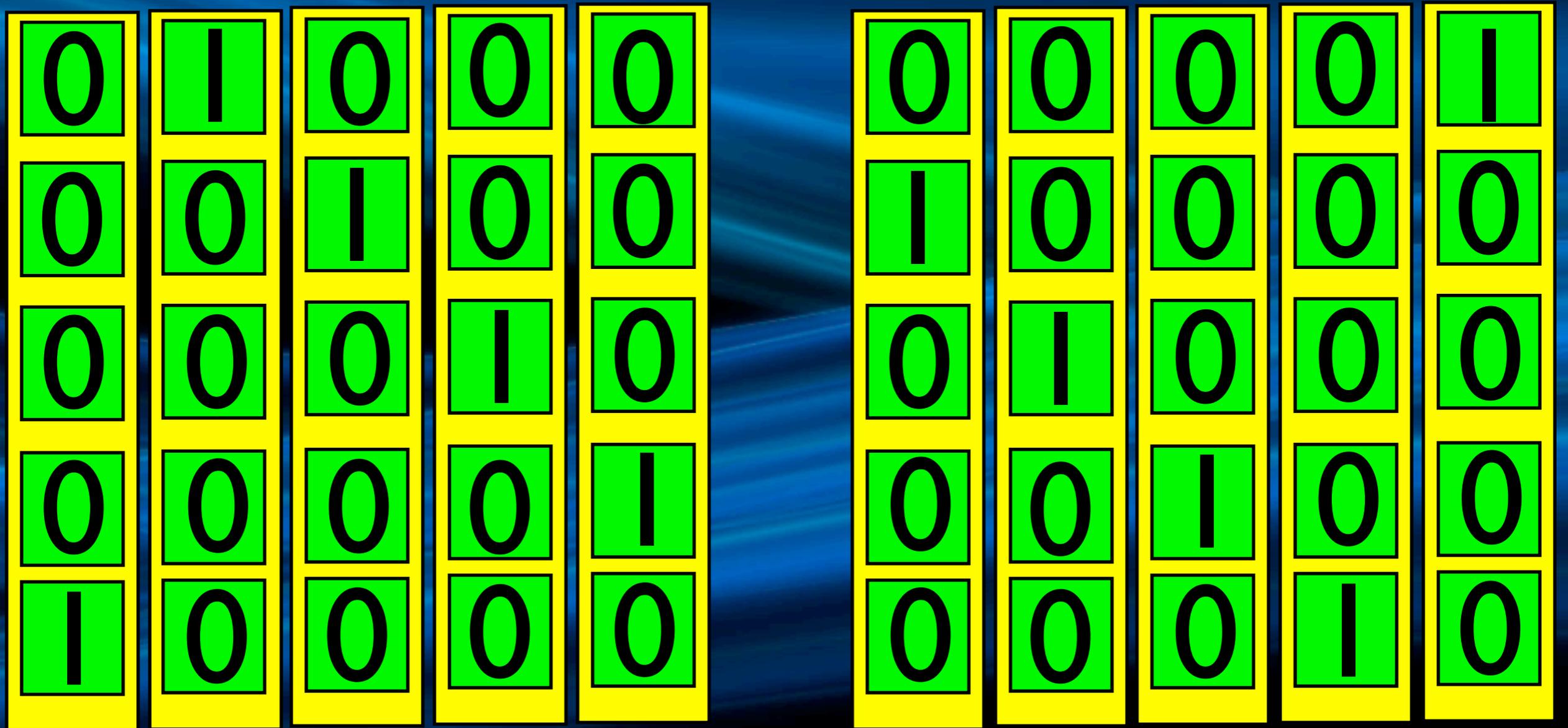
1)	Both have simple eigenvalues and the spectrum is the same
2)	Algebraic and geometric multiplicities are the same for A^n, B^n for all n
3)	Both are diagonalizable and have same eigenvalues.

Similar?

3	2	0	0
0	4	0	0
7	7	5	0
1	1	7	4

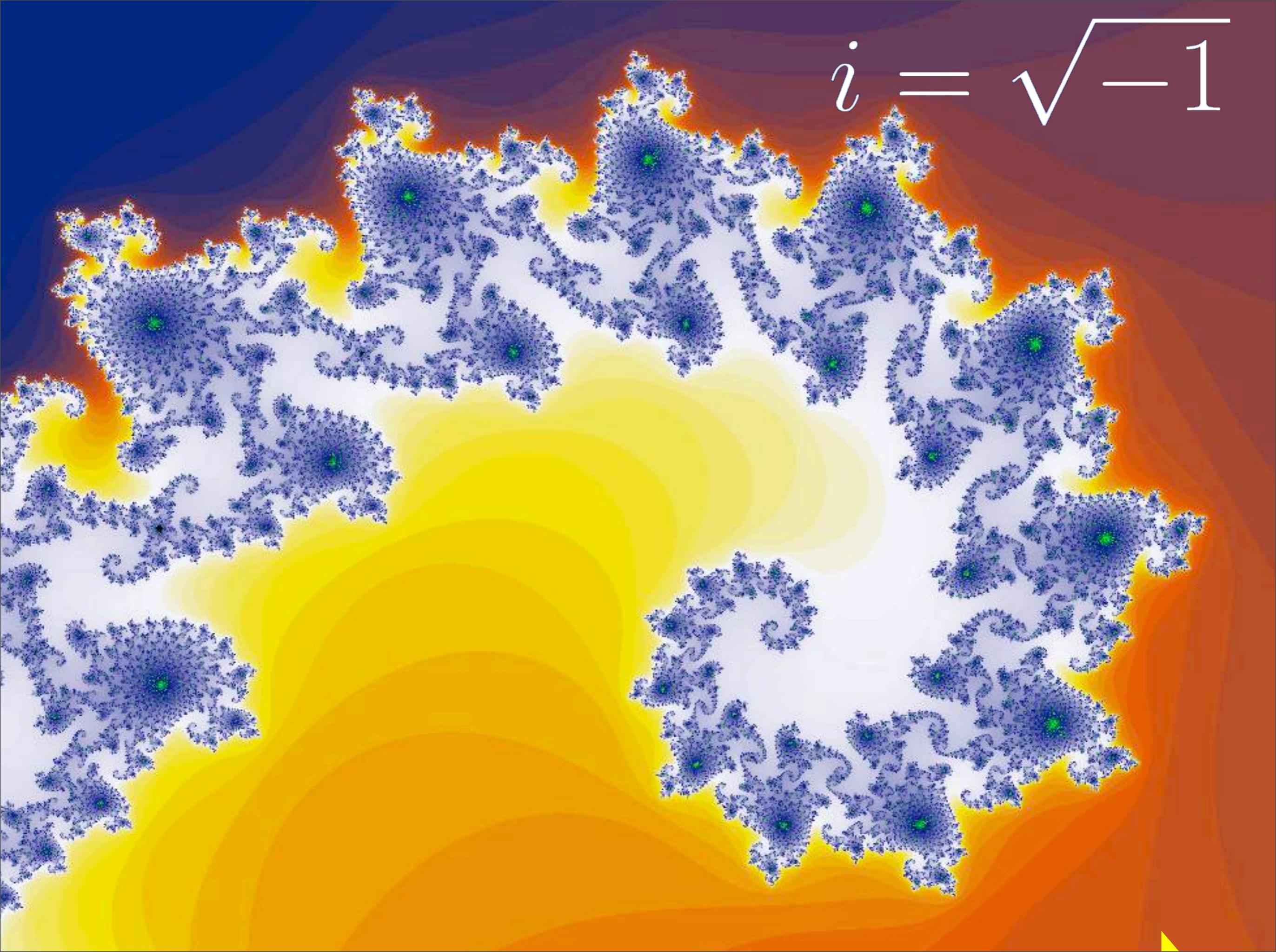
4	2	2	2
0	5	3	3
0	0	4	0
0	0	2	3

Similar?



8. Complex Numbers

$$i = \sqrt{-1}$$



Complex Numbers

$$i = \sqrt{-1}$$



Gauss in 1825 : “The true metaphysics of the square root of -1 is elusive”.

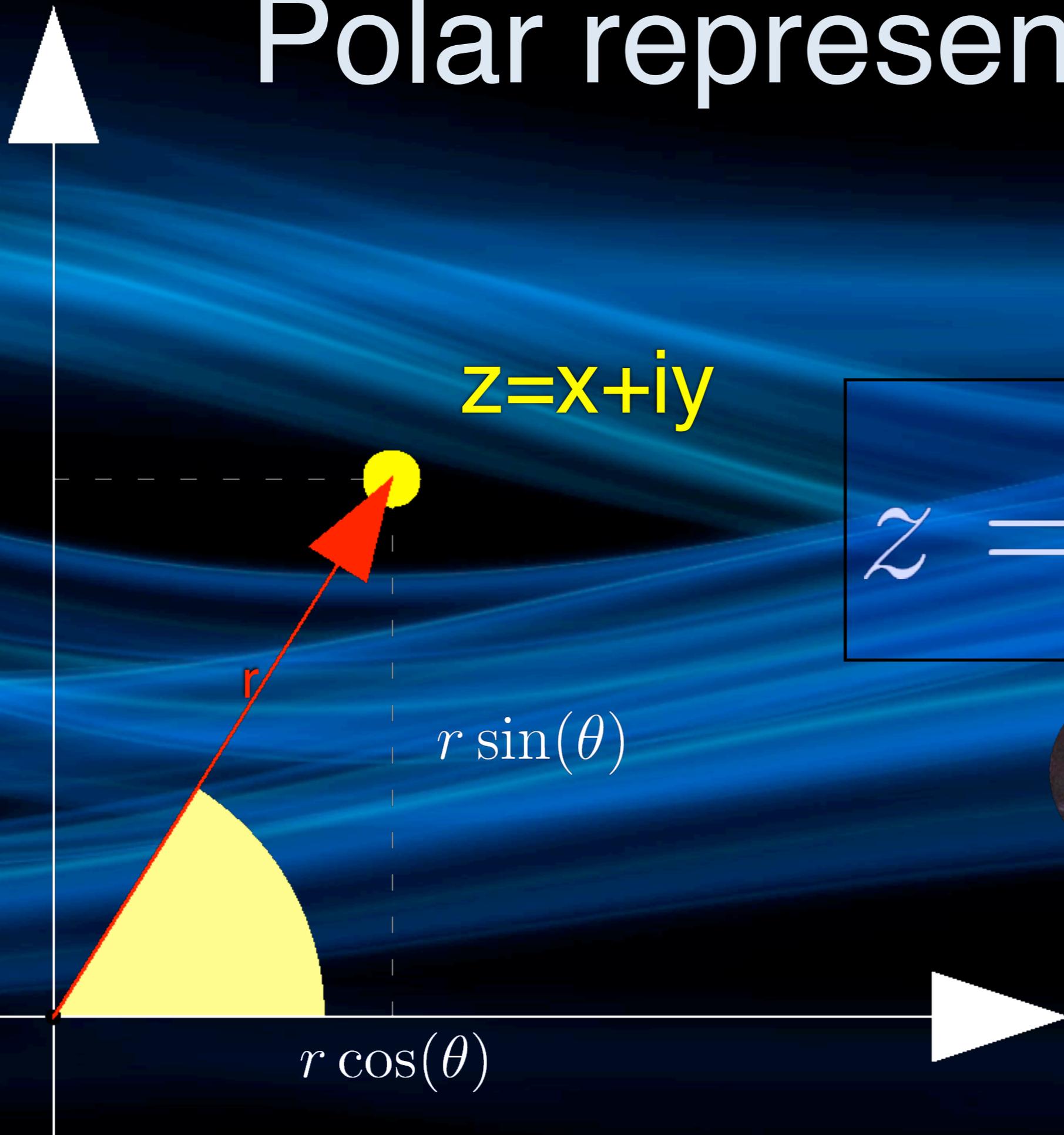
Euler Formula

$$\cos(\theta) + i \sin(\theta) = e^{i\theta}$$



Is the gateway to most secrets in complex numbers.

Polar representation



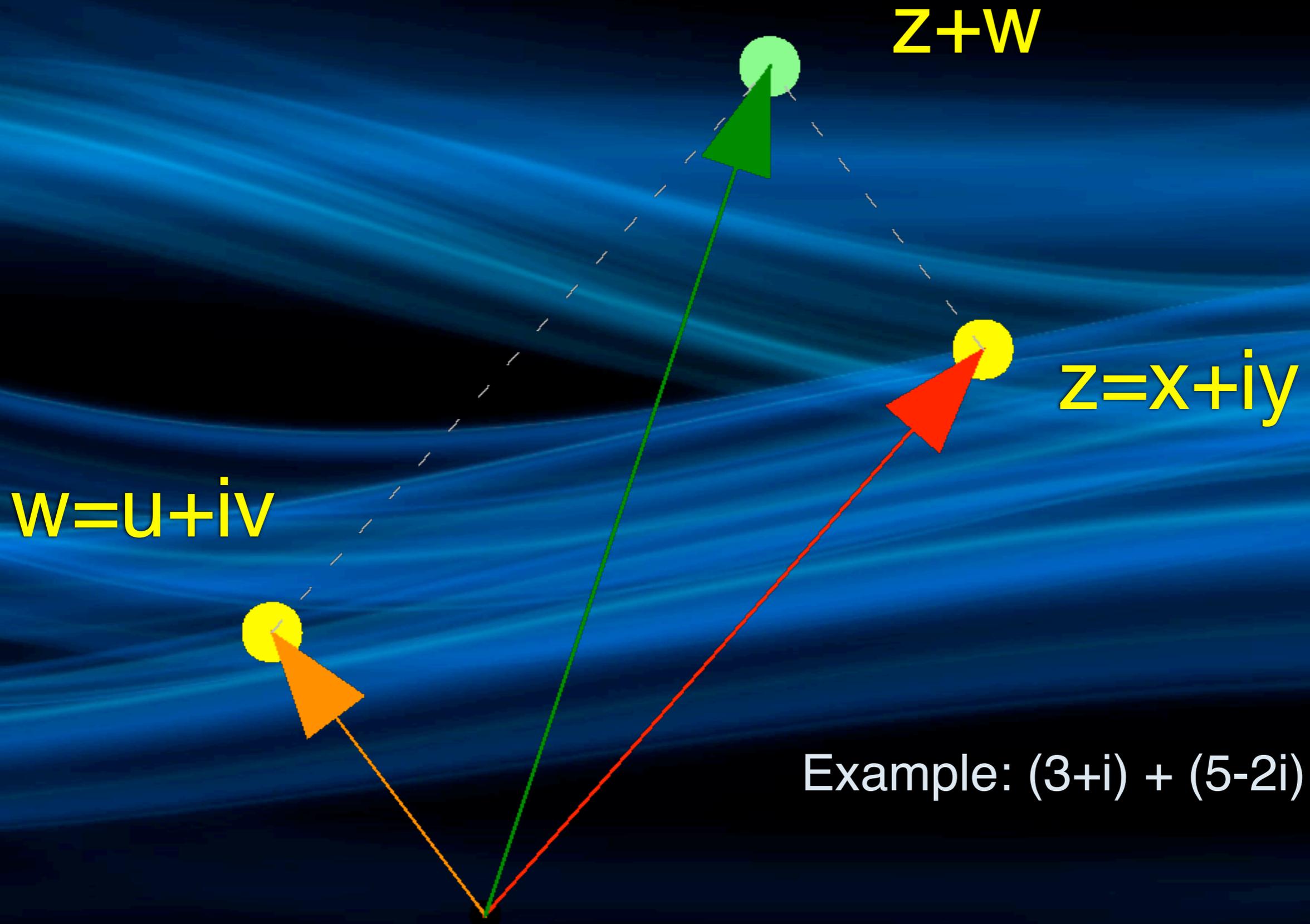
$$z = x + iy$$

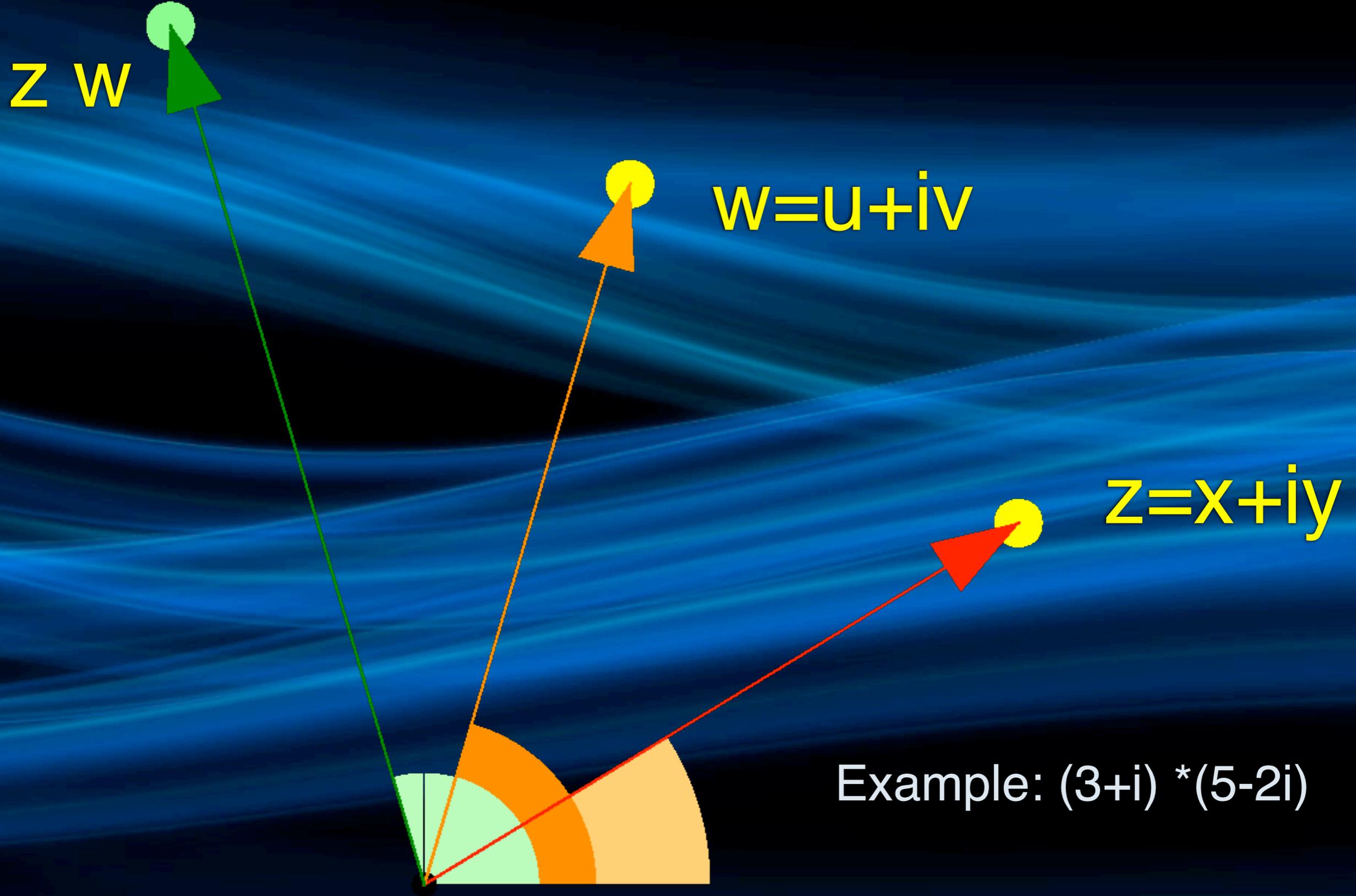
$$z = r e^{i\theta}$$



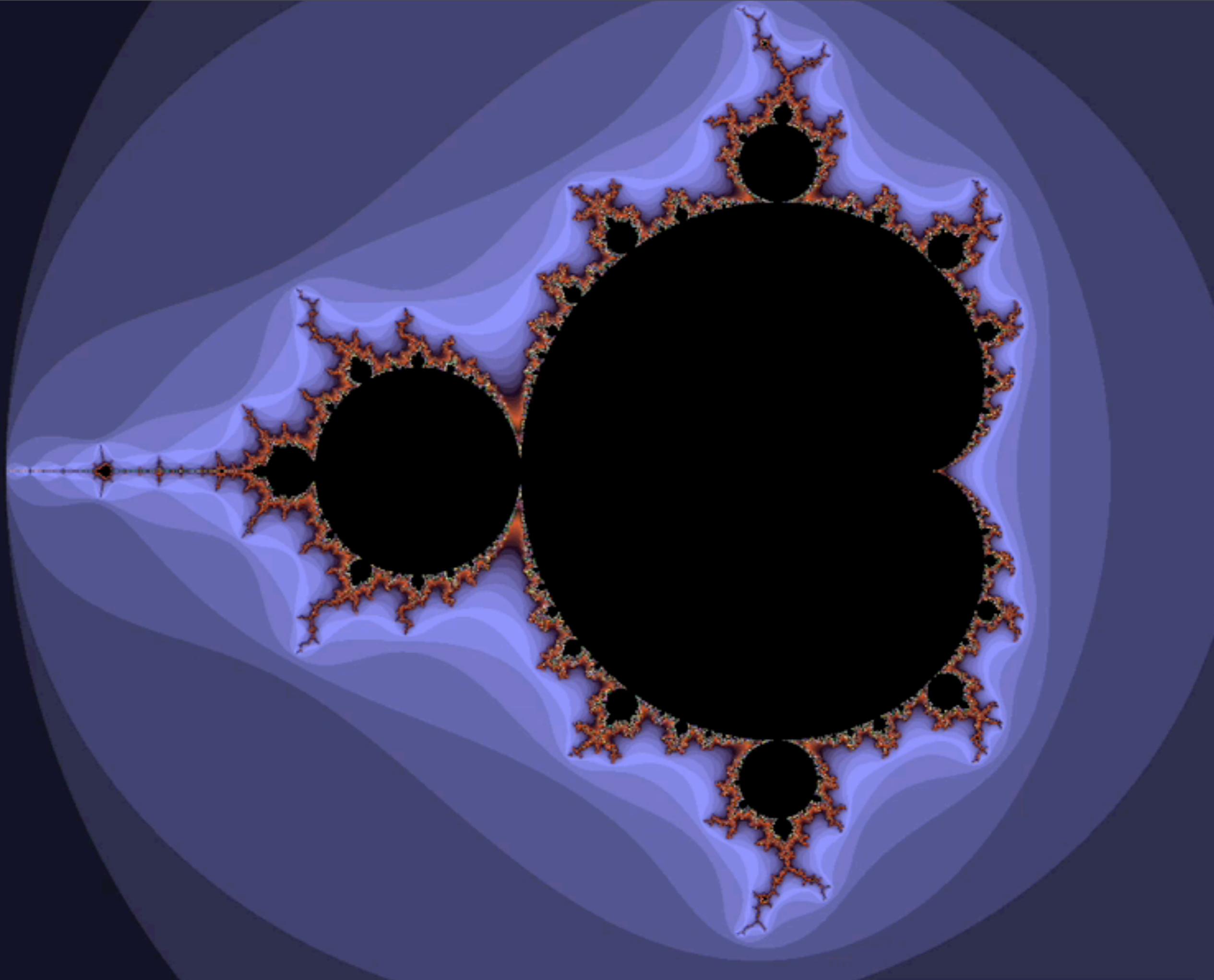
Gauss Plane

Addition

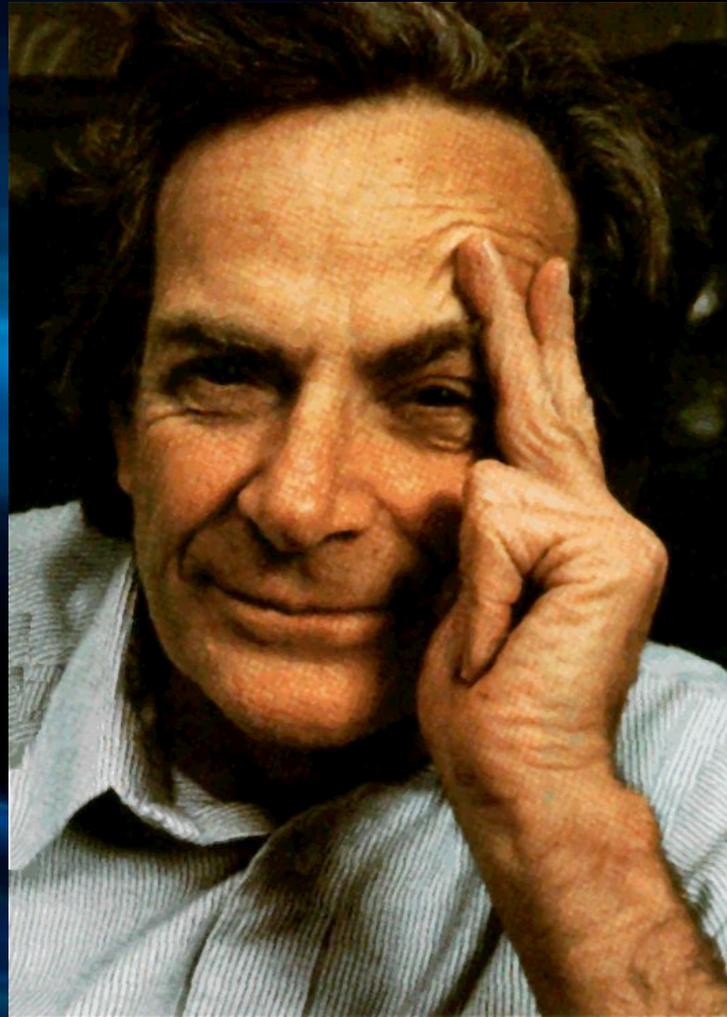




Example: $(3+i) * (5-2i)$

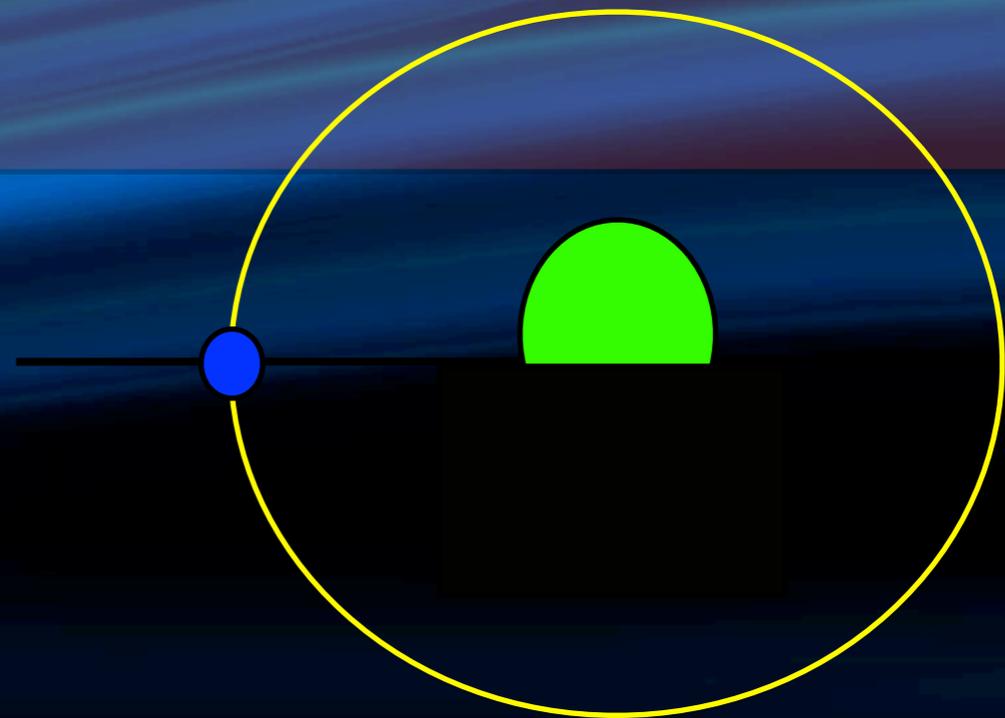


“The most remarkable
formula in math”

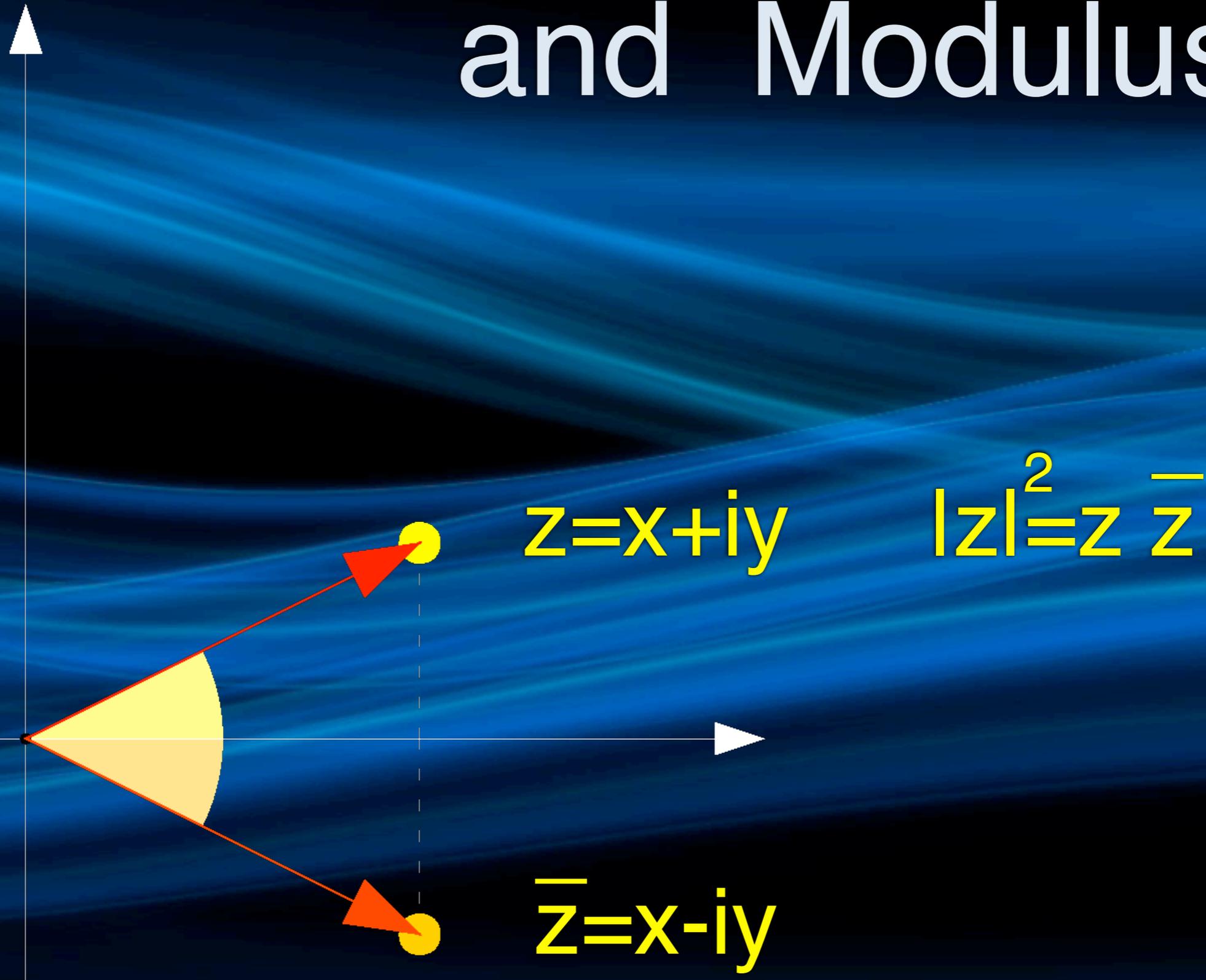


feynman

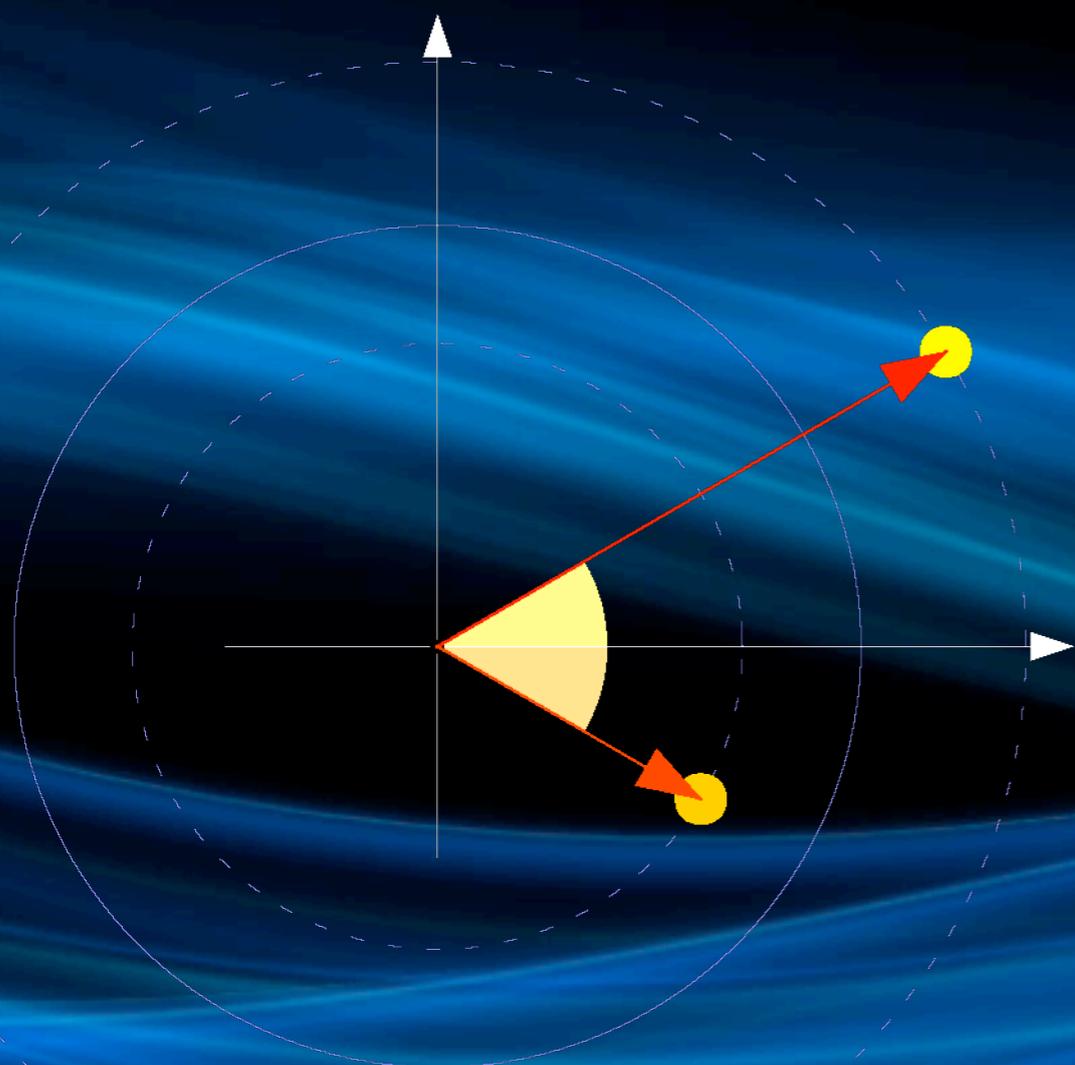
$$1 + e^{i\pi} = 0$$



Complex Conjugate and Modulus



Division



$$\frac{z}{w} = \frac{z\overline{w}}{w\overline{w}} = \frac{z\overline{w}}{|w|^2}$$

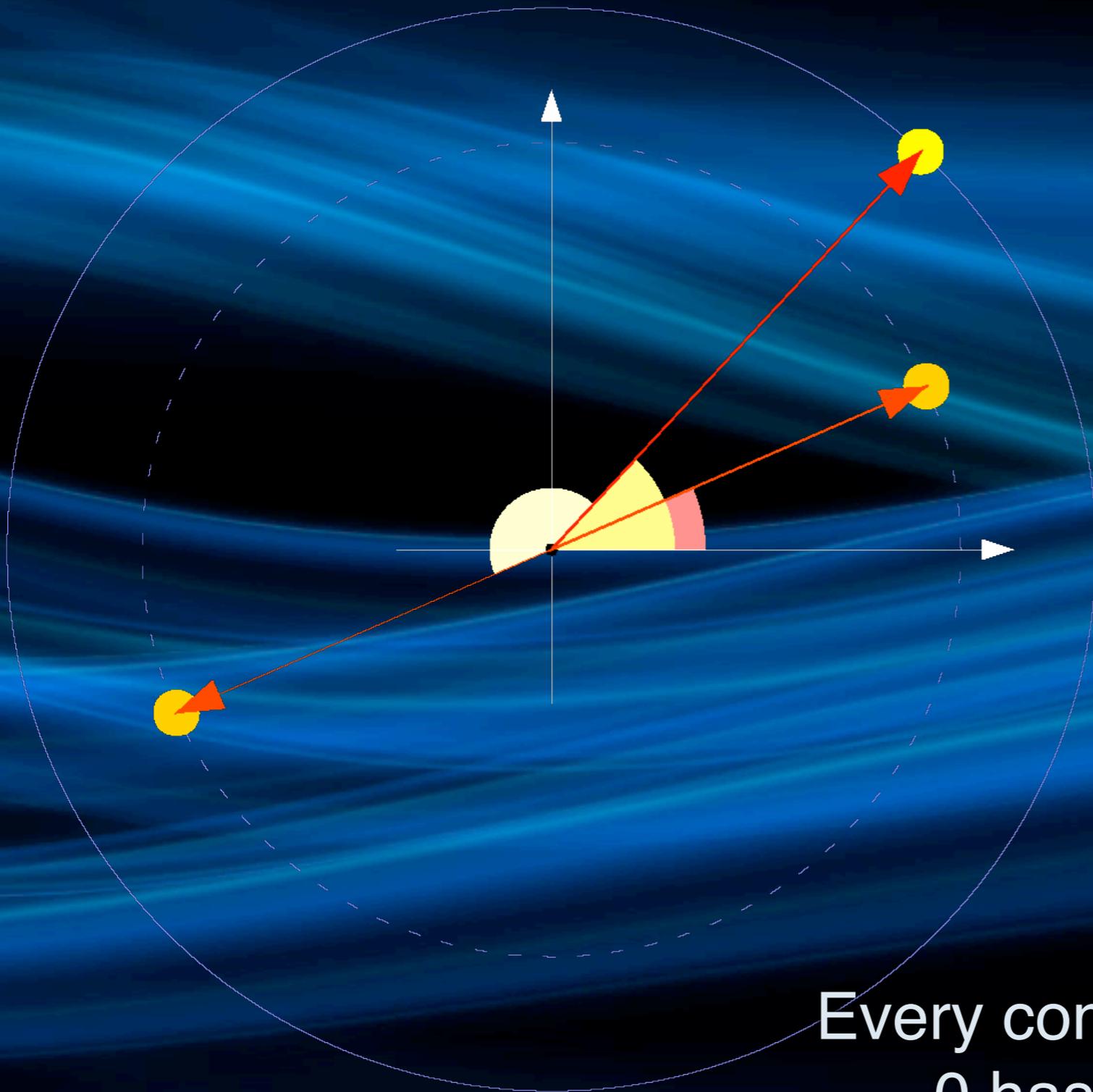
Fundamental theorem of algebra

$$p(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n$$

A polynomial of degree n
has
exactly n roots $p(x)=0$

especially the characteristic polynomial.

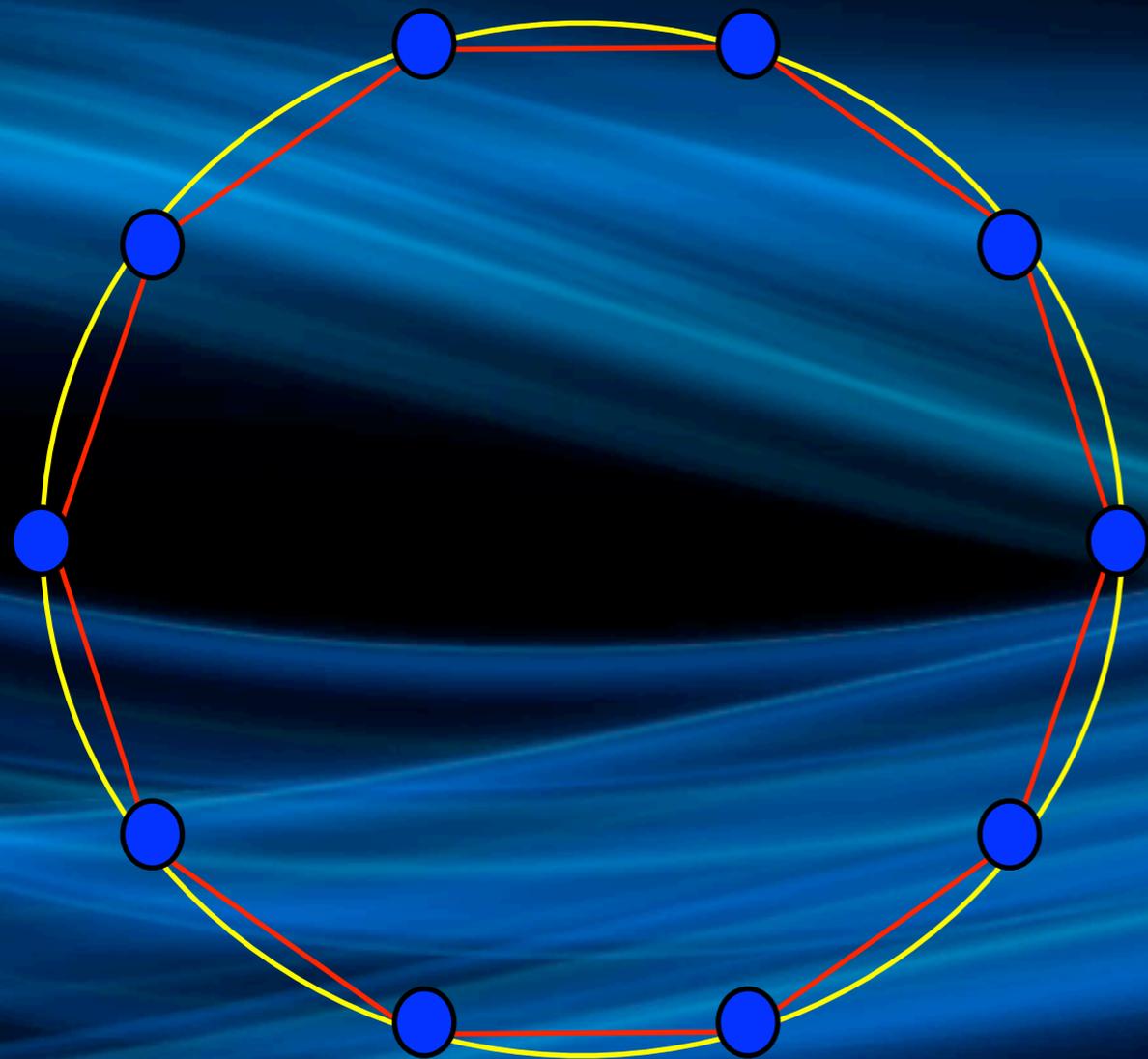
The square root



“Take square root of modulus and divide angle by 2”.

Every complex number different from 0 has exactly 2 square roots.

Higher roots

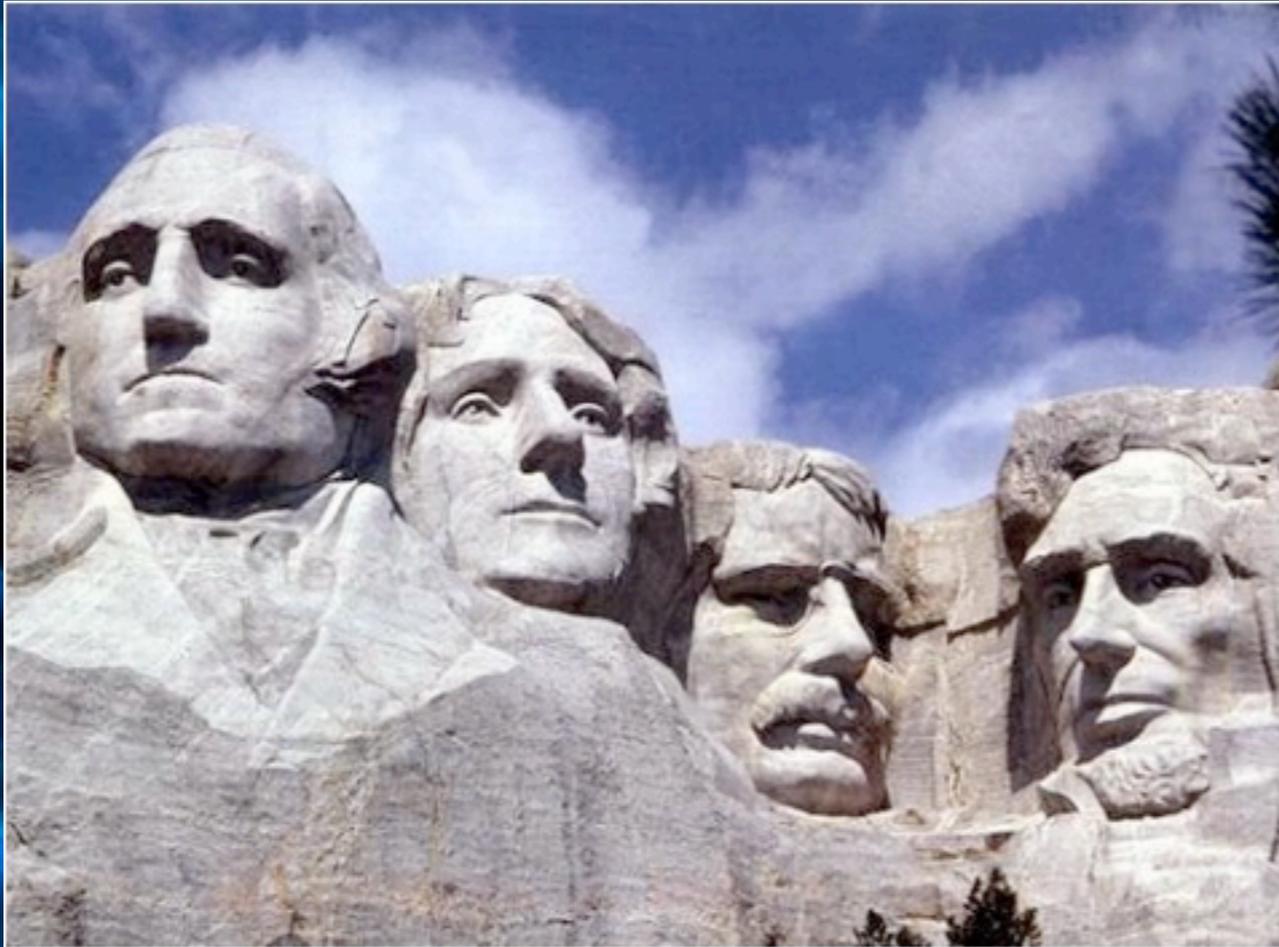


10

$2\pi ki$

$$z = r e$$

$$z = r^{1/n} e^{2\pi ki/n}$$



It can be written in the form $a + bi$, where a and b are real numbers, and

qwiki