Math 21b Review Sheet

(I) Orthonormal Bases and Orthogonal Transformations

ORTHONORMAL BASES of a ~~sub~~space are useful.

An orthonormal set of vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ has the properties

1) every vector has length 1 $(\vec{v}_i \cdot \vec{v}_i = 1)$

2) all vectors are perpendicular to each other $(\vec{v}_i \cdot \vec{v}_j = 0 \text{ if } i \neq j)$

Say we want to find the length of a given vector. If we can write it in terms of orthonormal basis vectors, it's an easy task:

$$\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_n\vec{v}_n$$

$$\begin{aligned}\vec{x}\cdot\vec{x} = {} & c_1^2\,\vec{v}_1\cdot\vec{v}_1 + c_1c_2\,\vec{v}_1\cdot\vec{v}_2 + c_1c_3\,\vec{v}_1\cdot\vec{v}_3 + \ldots + c_1c_n\,\vec{v}_1\cdot\vec{v}_n \\ & + c_1c_2\,\vec{v}_2\cdot\vec{v}_1 + c_2^2\,\vec{v}_2\cdot\vec{v}_2 + c_2c_3\,\vec{v}_2\cdot\vec{v}_3 + \ldots + c_2c_n\,\vec{v}_2\cdot\vec{v}_n \\ & + \ldots \\ & + c_1c_n\,\vec{v}_n\cdot\vec{v}_1 + c_2c_n\,\vec{v}_n\cdot\vec{v}_2 + \ldots + c_n^2\,\vec{v}_n\cdot\vec{v}_n\end{aligned}$$

Since the vectors are orthonormal, this ugly mess reduces to

$$\vec{x}\cdot\vec{x} = c_1^2 + c_2^2 + \ldots + c_n^2$$

$$\text{so } \|\vec{x}\| = \sqrt{\vec{x}\cdot\vec{x}} = \sqrt{c_1^2 + \ldots + c_n^2}$$

Or let's say we want to find the orthogonal projection of a vector $\vec{x}$ onto a subspace V of $\mathbb{R}^n$ — that is, a vector $\vec{w}$ perpendicular to every vector in V s.t. $\vec{x} = \vec{v} + \vec{w}$, $\vec{v}$ in V

(think of orthogonal projections onto lines or planes)

If we have an orthonormal basis for V, the formula is

$$\text{proj}_V \vec{x} = (\vec{x}\cdot\vec{v}_1)\vec{v}_1 + (\vec{x}\cdot\vec{v}_2)\vec{v}_2 + \dots + (\vec{x}\cdot\vec{v}_m)\vec{v}_m$$

where $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m$ is the orthonormal basis.

To find an orthonormal basis, we start with an ordinary basis and use Gram-Schmidt. If we start with $\vec{v}_1, \dots, \vec{v}_n$, we

1) first find an orthonormal basis of $V_1 = \text{span}(\{\vec{v}_1\})$
   (just normalize $\vec{v}_1$)
2) next find an orthonormal basis of $V_2 = \text{span}(\{\vec{v}_1, \vec{v}_2\})$
   (normalize $\vec{v}_2 - \text{proj}_{V_1}\vec{v}_2$)
3) find an orthonormal basis of $V_3 = \text{span}(\{\vec{v}_1, \vec{v}_2, \vec{v}_3\})$
   (normalize $\vec{v}_3 - \text{proj}_{V_2}\vec{v}_3$)

and so on.

If we run through this process, we can encode it as

$$A = QR$$

A's columns are the original basis vectors
Q's columns are the orthonormal basis vectors
R is an upper-triangular matrix relating A and Q

You might want to ~~~~ find the orthogonal complement of a subspace V. $V^\perp$ consists of those vectors which are perpendicular to every vector in V. Notice that if we have an orthonormal basis of V and an orthonormal basis of $V^\perp$, we can put them together to get an orthonormal basis of $\mathbb{R}^n$.

$T: \mathbb{R}^n \to \mathbb{R}^n$

An orthogonal transformation preserves lengths: $\|T\vec{x}\| = \|\vec{x}\|$ if T is orthogonal. An orthogonal transformation also preserves angles and dot products.

The columns of an orthogonal matrix are orthonormal basis vectors of $\mathbb{R}^n$ (this is because the columns are $T\vec{e}_1, T\vec{e}_2, \ldots, T\vec{e}_n$)

The inverse of an orthogonal transformation A is A^T.

The formula for $\text{proj}_V \vec{x}$ can also be written as

$\text{proj}_V \vec{x} = AA^T \vec{x}$, where the columns of A are the orthonormal basis vectors of V.

(II) Determinants

You should be able to compute the determinant of a matrix using patterns.

Another way to calculate the determinant is to use Gauss-Jordan elimination. That process depends on some properties of the determinant:

1 - Switching rows changes the sign

2 - The determinant is linear in its rows, which means

- Multiplying 1 row by k multiplies det by k
- This property: $\det\begin{bmatrix} 1 & 3 & 2 \\ 9 & 0 & 5 \\ 3 & 7 & 8 \end{bmatrix} = \det\begin{bmatrix} 1 & 2 & 1 \\ 9 & 0 & 5 \\ 3 & 7 & 8 \end{bmatrix} + \det\begin{bmatrix} 0 & 1 & 1 \\ 9 & 0 & 5 \\ 3 & 7 & 8 \end{bmatrix}$

1&2 lead to 3 - If two rows are equal, det $= 0$.

4 - You can add a multiple of 1 row to another without changing det.

$\det(A) = 0 \iff A$ is not invertible.

Some other properties:

$\det(A^T) = \det(A)$, so it doesn't matter if we consider the rows or columns

$\det(AB) = \det(A)\det(B)$

$\det(A^{-1}) = 1/\det(A)$

The determinant also has geometric properties:

- the volume of the k-parallelipiped spanned by $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ in $\mathbb{R}^n$ is $\sqrt{\det(A^T A)}$, where A's columns are $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$

  If $n=k$, the volume is just $|\det(A)|$

- the expansion factor of a linear transformation A is $|\det(A)|$

Cramer's Rule makes use of determinants:

- the system $A\vec{x} = \vec{b}$ has solution $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

  where $x_i = \dfrac{\det(A_i)}{\det(A)}$ if A is invertible.

  A_i is A with the ith column replaced by $\vec{b}$.

A corollary to Cramer's Rule:

$A^{-1} = \dfrac{1}{\det(A)} \operatorname{adj}(A)$, where $\operatorname{adj}(A)$ has i,j-entry equal to $\det(A_{ji})$, $A_{ji} = A$ with ith col and jth row removed.

III Eigenvalues and Eigenvectors

Say A is an $n \times n$ matrix and $\vec{v} \in \mathbb{R}^n$.

In all likelihood, it would take a lot of computation to calculate $A^6 \vec{v}$. However, if $A\vec{v} = \lambda \vec{v}$, λ a number, then $A^6 \vec{v} = \lambda^6 \vec{v}$. λ^6 is pretty easy to calculate.

If $A\vec{v} = \lambda \vec{v}$, $\vec{v} \neq \vec{0}$, $\vec{v}$ is an eigenvector of A and λ is an eigenvalue of A.

To find the eigenvalues of A, look for λ s.t.
$\lambda I_n \vec{v} = A\vec{v} \Rightarrow (\lambda I_n - A)\vec{v} = \vec{0}$ for some $\vec{v} \neq \vec{0}$;
this is the same as finding λ s.t.

$$\det(\lambda I_n - A) = 0$$

since the determinant is 0 iff the matrix is not invertible

$\det(\lambda I_n - A)$ is an nth-degree polynomial $f_A(\lambda)$, the characteristic polynomial of A. Its roots are the eigenvalues. The algebraic multiplicity of λ_0 is the highest power of $\lambda - \lambda_0$ that divides $f_A(\lambda)$.

Once we find the eigenvalues, we need to find the corresponding eigenspaces.

$$E_{\lambda_0} = \{ \vec{v} : A\vec{v} = \lambda_0 \vec{v} \} = \ker(\lambda_0 I_n - A)$$

The dimension of E_{λ_0} as a subspace of $\mathbb{R}^n$ is the geometric multiplicity of λ_0. The geometric multiplicity is $\leq$ the algebraic

Even if $\vec{v}$ is not an eigenvector of A, it is not difficult to calculate $A^n\vec{v}$ if we can write $\vec{v}$ as a linear combination of eigenvectors:

$$\text{if } \vec{v} = c_1\vec{v}_1 + \dots + c_m\vec{v}_m, \quad A\vec{v}_i = \lambda_i\vec{v}_i$$
$$\text{then } A^n\vec{v} = c_1\lambda_1^n\vec{v}_1 + c_2\lambda_2^n\vec{v}_2 + \dots + c_m\lambda_m^n\vec{v}_m$$

This is why eigenbases - bases of eigenvectors - are so nice. A has an eigenbasis if the geometric multiplicities of its eigenvalues add up to n.

Eigenvalues can also be complex numbers. The process for finding complex eigenvalues and eigenvectors is the same as above.

Tr(A) = Sum of complex eigenvalues of A

Det (A) = Product of complex eigenvalues of A

We are often concerned with the long-term behavior of a dynamical system modeled by $\vec{x}(t) = A^t\vec{x}(0)$. If the complex eigenvalues all have modulus <1, $\vec{x}(t)$ will tend to 0; otherwise, no. In the 2x2 case, ~~the~~ if there are 2 complex eigenvalues, the trajectory spirals inward, outward, or on an ellipse if the modulus is <1, >1, $=1$ respectively.

Bridget Tenner

A bit of review for Math 21b Midterm 2 on Monday, November 22, 1999

(This review covers through section 6.4, but does not include the final section or two that may be on the midterm)

Things to keep in mind

Dot product – If $v = (v_1, v_2, \ldots, v_n)$ and $w = (w_1, w_2, \ldots, w_n)$, then their dot product, $v \cdot w$, is equal to $v_1w_1 + v_2w_2 + \ldots + v_nw_n$

Orthogonality – two vectors are orthogonal (perpendicular) if and only if their dot product is zero

Length – the length of a vector v is $||v|| = \sqrt{(v \cdot v)}$; the unit vector has length 1

Triangle inequality -- $||v + w|| \leq ||v|| + ||w||$

Orthonormal vectors – a set of unit vectors that are all perpendicular; they are all linearly independent and form a basis; another way to write this is to say $v_i \cdot v_j = 1$ if $i = j$, and $v_i \cdot v_j = 0$ if $i \neq j$

Orthonormal basis – a set of orthogonal unit vectors $v_1, \ldots, v_n$ that form a basis for the space (are linearly independent and span it); if these vectors span Rn, for any x in Rn, $x = (v_1 \cdot x)v_1 + \ldots + (v_m \cdot x)v_m$, and this the unique way to write x as a linear combination of $v_1, \ldots, v_n$

Orthogonal complement of space V – denoted $V^\perp$; the set of vectors in $\mathbf{R}^n$ (where V is a subset of $\mathbf{R}^n$) that are orthogonal to all vectors in V; it is a subspace if V is a subspace; another way to write this is $\{x \text{ in } \mathbf{R}^n \mid v \cdot x = 0 \text{ for all } v \text{ in } V\}$

Orthogonal projection onto a subspace V with orthonormal basis $v_1, \ldots, v_m$ – for all x in $\mathbf{R}^n$, there exists a unique w in V such that $(w - x)$ is in $V^\perp$; w is the orthogonal projection of x onto V; the formula, which is a linear transformation, for this is $\text{proj}_V(x) = (v_1 \cdot x)v_1 + \ldots + (v_m \cdot x)v_m$

Cauchy Schwartz inequality – $|x \cdot y| \leq ||x||\,||y||$

Angle – the angle, α, between vectors x and y is equal to $\arccos[(x \cdot y) / (||x||\,||y||)]$

Gram-Schmit Process – a way of making a set of linearly independent vectors into an orthonormal set of vectors without changing the actual space they span; the steps in the process can be found on page 201 of the text and basically amount to the repetition of two steps for each vector:

1. Make the vector perpendicular to all the other vectors already looked at
2. Make it a unit vector

QR Factorization – a way of storing the information used (lengths and projection) in the Gram-Schmit Process in two matrices; the exact content of each matrix can be found on page 202 of the text

Transpose – the transpose of a matrix reflects the elements across the main diagonal; in other words, elements a_{ij} and a_{ji} switch with each other; notice that we can now write $v \cdot w$ as v^Tw; the transpose has the following properties:

1. $(AB)^T = B^TA^T$ (note that the order switches!)
2. $(A^T)^{-1} = (A^{-1})^T$
3. rank A = rank A^T

Symmetric – a matrix A is called symmetric if $A^T = A$

Skew-symmetric – a matrix A is called skew-symmetric if $A^T = -A$

Orthogonal Transformation – a transformation T from $\mathbf{R}_n$ to $\mathbf{R}_n$ that preserves length and angles; T is orthogonal if and only $T(e_1), \ldots, T(e_n)$ are an orthonormal basis of $\mathbf{R}_n$; this is equivalent to saying that A is an orthogonal matrix if and only if its columns are an orthonormal basis; products and inverses of orthogonal matrices are orthogonal; the following statements about A an nxn matrix are equivalent (either they are all true or they are all false):

1. A is an orthogonal matrix
2. A(x) preserves length: $||A(x)|| = ||x||$
3. The columns of A form an orthonormal basis for $\mathbf{R}^n$
4. $A^TA = I_n$
5. $A^{-1} = A^T$

Orthogonal projection onto V – if V is a subspace of $\mathbf{R}_n$ with orthonormal basis $v_1, \ldots, v_m$ and A is the matrix whose columns are $v_1, \ldots, v_m$, then the matrix representing the orthogonal projection onto V is AA^T (note the order)

Determinant – the determinant of a matrix (only square matrices have determinants, it is not defined for a nonsquare matrix) can be thought of in two ways: as a map from the set of square matrices to the real numbers, or as a real number associated with a matrix; the more correct way to view it is probably as a map since, but either way is acceptable

Determinant of a 1x1 matrix – if the matrix is [a] then the determinant is just a

Determinant of a 2x2 matrix – if the matrix has columns (a, c) and (b, d) then the determinant is ad-bc

Determinant of a 3x3 matrix – if the columns of the matrix are u, v, and w, then the determinant is equivalent to $u \cdot (v \times w)$; another way to find the determinant is described on page 240 of the text

Pattern – a pattern is the product of n entries of a nxn matrix such that exactly one is in each column and one is in each row

Inversion – an inversion occurs when one element of a pattern is below and to the left of another element of a pattern

Determinant of an nxn matrix – the determinant of a matrix = sum(patterns with even number of inversions) – sum(patterns with odd number of inversions); det (A^T) = det (A); the determinant is linear in the columns (and hence in the rows as well), in other words, if $v_1, \ldots, v_{j-1}, v_{j+1}, \ldots, v_n$ are vectors in $\mathbf{R}^n$ and T(x) is a function which takes the determinant of a matrix whose columns are $v_1, \ldots, v_{j-1}, x, v_{j+1}, \ldots, v_n$ then this is a linear transformation: $T(x+y) = T(x) + T(y)$ and $T(kx) = kT(x)$ where k is any scalar

Effects of row operations on the determinant: A is an nxn matrix and B is the matrix obtained by performing some operation on A; if the operation is swapping two rows, det (B) = -det (A); if the operation is dividing a row of A by k, then det (B) = (1/k) det (A); if the operation is adding a scalar multiple of one row to another, then det (B) = det (A); this can be summarized by saying that in the process of row reducing A to the identity matrix, if s = number of swaps (of rows) made and $k_1, \ldots, k_r$ are the scalars that individual rows were divided by, then det (A) = $(-1)^s k_1 k_2 \cdots k_r$

An nxn matrix A is invertible if and only if det (A) $\neq 0$

det (AB) = det (A) det (B); note, the determinant of the sum of two matrices is not necessarily equivalent to the sum of the determinants of two matrices

det $(A^{-1}) = (\det(A))^{-1} = 1/(\det(A))$

Minor – if A is an nxn matrix, then A_{ij} is the (n-1)x(n-1) matrix formed by eliminating the ith row of A and the jth column of A

Another way to evaluate the determinant of A – the determinant of an nxn matrix can also be found by det (A) = a_{11} det $(A_{11}) - a_{12}$ det $(A_{12}) + \ldots + (-1)^{1+n} a_{1n}$ det (A_{1n}) or any similar such expansion about any row or column; the exact formulas for this can be found on page 262 of the text

Determinant of an orthogonal matrix is ± 1; if it is 1 then the matrix is a rotation matrix

Parallelepiped – if $v_1, \ldots, v_n$ are vectors in $\mathbf{R}^n$, then the area of the parallelepiped defined by them is the determinant of the matrix whose column vectors are $v_1, \ldots, v_n$; if you only have vectors $v_1, \ldots, v_k$ in $\mathbf{R}^n$ where $k \leq n$, the volume of the k-parallelepiped is $\sqrt{}(\det(A^TA)$ where A is the matrix whose columns are $v_1, \ldots, v_k$; and in the case where $k = n$, this is exactly the same as the first definition

Yet another way to find a determinant – if $v_1, \ldots, v_n$ are the columns of matrix A, then $|\det(A)| = \|v_1\| \|v_2 - \text{proj}_{V_1} v_2\| \cdots \|v_n - \text{proj}_{V_{n-1}} v_n\|$ where V_i is the space spanned by vectors $v_1, \ldots, v_i$; note that this says $|\det(A)| = |\det(R)|$ where R is the matrix from the QR factorization of A

Expansion factor – if Ω is some region and T is a linear transformation represented by the matrix A, (area of $T(\Omega)$)/(area of Ω) = $|\det(A)|$

Cramer's Rule: to solve the system $Ax = b$, the ith coordinate of the vector x, $x_i = \det(A_i)/\det(A)$ where A_i is the matrix formed by replacing the ith column of A by the vector b

State vector – a vector that contains all the information of what is happening in a system at a time t (see Roadrunner example on page 291)

Phase portrait – a picture of what happens at present, past, and future times; it plots the vectors x(t) for all integers t (see Roadrunner example on page 291)

Eigen vector – an eigen vector of the matrix A is a nonzero vector v such that $A(v) = \lambda v$ for some value λ

Eigen value – the value λ referred to above

Eigen values of an orthogonal matrix – the possibilities are ± 1

Very important to study – go through Summary 6.1.4 on page 302; memorize it if necessary, but make sure you understand roughly why each of the statements are equivalent

Characteristic polynomial – $f_A(\lambda) = \det(\lambda I_n - A)$; the eigen values of an nxn matrix A are the zeroes of this function; λ is an eigen value of A if and only if $\det(\lambda I_n - A) = 0$

Eigen values of a triangular matrix – the diagonal entries

Trace – the sum of the diagonal elements of an nxn matrix A is called the trace of A, written tr (A)

Eigen values of a 2x2 matrix – $f_A(\lambda) = \lambda^2 - \text{tr}(A)\lambda + \det(A)$

Algebraic multiplicity –eigen value λ_0 has algebraic multiplicity k if $f_A(\lambda) = (\lambda - \lambda_0)^k g(\lambda)$ for a polynomial $g(\lambda)$ that λ_0 is not a root of; in other words, λ_0 is a root of multiplicity k of $f_A(\lambda)$

An nxn matrix has at most n eigen values, even if they are counted with their algebraic multiplicities; if n is odd, then an nxn matrix has at least one eigen value

Eigen space – the kernel of the matrix $\lambda I_n - A$ is the eigen space associated with λ and written E_λ; this space consists of all solutions v to the equation $A(v) = \lambda v$

Geometric multiplicity – if λ is an eigen value of matrix A, then the dimension of E_λ is the geometric multiplicity of λ; (geometric multiplicity of λ) = $\dim(E_\lambda) \leq$ (algebraic multiplicity of λ)

Eigenbasis – if A is an nxn matrix, a basis of $\mathbf{R}^n$ consisting of eigen vectors of A is an eigenbasis for A

If $v_1, v_2, \ldots, v_m$ are eigen vectors of an nxn matrix A with distinct eigen values $\lambda_1, \lambda_2, \ldots, \lambda_m$, then the vectors are linearly independent; if A has n distinct eigen values then there is an eigenbasis for A found by choosing an eigen vector for each eigen value; if the geometric multiplicities of the eigen values of A add up to n then there is an eigenbasis for A found by choosing a basis of each eigen space and combining these vectors

Complex numbers – $z = a + ib$, where $i = \sqrt{}(-1)$; a is called the "real part" of z ($a = \text{Re}(z)$) and b is called the "imaginary part" of z ($b = \text{Im}(z)$); the length of z, $|z|$, is called the modulus of z and the angle z makes, ϕ, is called the argument of z; complex numbers operate in the following ways:

$$(a + ib) + (c + id) = (a + c) + i(b + d)$$
$$(a + ib)(c + id) = (ac - bd) + i(ad + bc)$$

Polar form – the polar form of the complex number z is $z = r(\cos\phi + i \sin\phi)$

$|zw| = |z|\,|w|$

$\arg(zw) = \arg z + \arg w \qquad (\text{modulo } 2\pi)$

DeMoivre's formula – this formula is: $(\cos\phi + i\sin\phi)^n = \cos(n\phi) + i\sin(n\phi)$

Fundamental theorem of algebra – any polynomial $p(\lambda)$ with complex coefficients can be written as a product of linear factors: $p(\lambda) = k(\lambda - \lambda_1)(\lambda - \lambda_2)\cdots(\lambda - \lambda_n)$ for some complex numbers (not necessarily distinct) $\lambda_1, \ldots, \lambda_n$, k; so a polynomial of degree n has exactly n roots when counted with multiplicity

Complex eigen values and eigen vectors – a complex nxn matrix has n complex eigen values if eigen values are counted with their algebraic multiplicities; if the eigen values are $\lambda_1, \ldots, \lambda_n$, listed with their algebraic multiplicities, then $\operatorname{tr}(A) = \lambda_1 + \ldots + \lambda_n$ and $\det(A) = \lambda_1 \cdots \lambda_n$

Practice Problems

(You certainly do not have to do all of these in order to be prepared for the midterm, but it would not be a bad idea to look through them. Also, all the problems are odd-numbered problems, so their answers should be in the back of the book)

Section 4.1: 5, 13, 17, 19, 27, 31
Section 4.2: 7, 21, 33, 35, 41
Section 4.3: 7, 13, 17, 25, 31
Section 5.1: 7, 19, 31, 39
Section 5.2: 3, 9, 17, 19, 25, 31, 39
Section 5.3: 7, 13, 21, 23, 27, 31
Section 6.1: 5, 17, 23, 27, 35, 39, 41
Section 6.2: 11, 15, 19, 27, 29, 37
Section 6.3: 5, 9, 19, 25, 35, 37, 41, 45
Section 6.4: 5, 11, 19, 23, 35, 37

More Questions/Problems

1. Consider the matrix A: $\begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$

   a. What are the eigen values and eigen vectors of A?
   b. If x is the vector (1, 1, 1), write x as a linear combination of the eigen vectors of A.
   c. Without technology, and without multiplying A by itself 7 times, what is $A^7(x)$?

2. Consider the matrix A: $\begin{bmatrix} 1 & 3 & 5 \\ 2 & 5 & 4 \\ -1 & -2 & 0 \end{bmatrix}$

solve the equation $A(x) = b$ for the following choices of b using Cramer's Rule

a. $b = (-1, -1, 0)$
b. $b = (-4, -2, 5)$

3. Find all real values of a so that the vectors $v_1 = (a, 1, -1)$, $v_2 = (0, 3a, -4)$, and $v_3 = (2, -1, 5)$ form a linearly independent set.

4. Given an nxn matrix A and a nonzero vector x in $\mathbf{R}^n$, complete the following:
   a. The nonzero vector x is called an eigenvector of A if ...
   b. The scalar discussed in (a) is called ...
   c. The eigenspace corresponding to a given scalar mentioned in (b) is ...

5. Given the matrix A: $\begin{bmatrix} 3 & 2 & 2 \\ 1 & 4 & 1 \\ -2 & -4 & -1 \end{bmatrix}$

   Write out the characteristic polynomial of A and find the eigen values. Find eigen vectors corresponding to each eigen value.