Mathematics 21b - Fall 2000 - First Exam

This exam was 90 minutes long. Calculators were not allowed.

1) (12 points) True or False. (Circle one) You need not give your reasoning.

a) The kernel of rref(**A**) is the same as the kernel of **A**. a) TRUE FALSE

b) The image of rref(**A**) is the same as the image of **A**. b) TRUE FALSE

c) Let **A** and **B** be n x n matrices, with **AB** = **BA**. Then $\mathbf{A}^3\mathbf{B} = \mathbf{B}\mathbf{A}^3$. c) TRUE FALSE

d) There is a 4 x 4 matrix **A** such that image(**A**) and kernel(**A**) are the same subspace of $\mathbf{R}^4$. d) TRUE FALSE

e) If **A** and **B** are n x n matrices such that the kernel of **A** is contained in the image of **B**, then the matrix **AB** cannot be invertible. e) TRUE FALSE

f) If **A** is an invertible 2 x 2 matrix, then the rank of **A** is necessarily 1. f) TRUE FALSE

2) (18 points) Let **A** be the 4 x 5 matrix $\mathbf{A} = \begin{bmatrix} 1 & -2 & 1 & 5 & 0 \\ 1 & -2 & -1 & 1 & 0 \\ 3 & -6 & -1 & 7 & -1 \\ -1 & 2 & 0 & -3 & 0 \end{bmatrix}$.

a) Represent the kernel of **A** as the span of a set of vectors.

b) Represent the image of **A** as the span of a set of vectors. Eliminate any vectors which can be expressed as linear combinations of other vectors in your set. [This is called a basis for im(**A**).]

c) Find all solutions of the equation **Ax** = **b**, where $\mathbf{b} = \begin{bmatrix} 3 \\ 5 \\ 1 \\ -4 \end{bmatrix}$

3) (16 points) Let S be a shear transformation in $\mathbf{R}^2$ along the x-axis that sends the vector $\mathbf{e}_2$ to the vector $2\mathbf{e}_1 + \mathbf{e}_2$. Let P be orthogonal projection onto the x-axis. Find matrices for P, S, and PS. What is $(PS)^{50}\begin{bmatrix} 22 \\ 17 \end{bmatrix}$?

4) (18 points) Given the linear system :

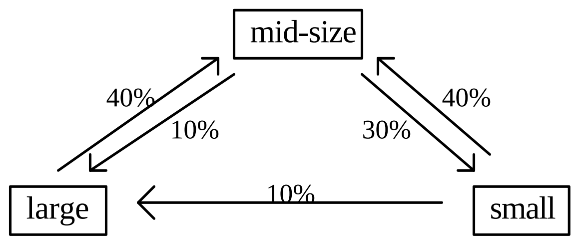
$$\begin{aligned} 2x + y - 2z &= 2 \\ -3x + 3y + 5z &= 7 \\ -x + 2y + 2z &= 1 \end{aligned}$$

a) Find the inverse of the matrix of coefficients **A**.

b) Check your answer to part (a) by calculating $\mathbf{A}^{-1}\mathbf{A}$.

c) Use $\mathbf{A}^{-1}$ to solve the given system of equations.

5) (18 points) A survey was conducted this year among 1000 drivers, comparing the sizes of the cars they were driving in 1990 with those they were driving today (all of them were driving in 1990). The results of the survey are represented in the diagram to the right. The diagram tells us, for example, that 40% of the people driving a larger car in 1990 had switched to a mid-size car by 2000. Those drivers that did not switch continued to drive the same size car.



Surprisingly, despite all of the changes, the survey showed that the numbers of people driving large, mid-size and small cars were exactly the same in 2000 as they had been in 1990. Find these numbers, and show your work.

6) (18 points) Let L be the line through the origin in $\mathbf{R}^3$ in the direction of $\begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$.

Let S be the orthogonal complement of L, i.e. the plane through the origin having this vector as a normal vector.

a) Find the matrix **A** of the orthogonal projection whose image is L.

b) What is the kernel of this matrix **A**?

c) Find the matrix of the orthogonal projection onto the plane S.
[Hint: If you let **p** be the vector projection of a given vector **x** in the direction of the normal to this plane, then the difference vector **x** - **p** will be the projection of **x** in the plane.]