1. [6 points] A matrix $M$ is of the form

$$\begin{bmatrix} 0 & * & 5 & * \\ * & * & * & * \end{bmatrix},$$

where as usual the $*$'s denote unknown and possibly different real numbers. Given that <u>$M$ is in row-reduced echelon form</u>, find all possible $M$, and explain why there are no other possibilities. For each of the $M$ that you have found, determine its rank, image, and kernel.

Given $M$ is 2x4. with $a_{11}=0$ and $a_{13}=5$. and $M$ in row-reduced form.

Since the first row is not all zeroes, it must have a leading 1 before the non-zero entry.

Hence $a_{12}=1$.

Every other entry in a column with a leading 1 must be zero.

Hence $a_{22}=0$.

If $a_{21}$ were non-zero, then we could divide the second row by that entry and have a leading 1, but it would be below and left of $a_{12}$, which is not row-reduced form.

Hence $a_{21}=0$.

If $a_{23}$ were non-zero, then it must be a leading 1 since it would be the first non-zero entry of the second row. But then $a_{13}$ would be zero since it is in the same column as a leading 1.

Hence $a_{23}=0$.

Thus far, we have $M = \begin{bmatrix} 0 & 1 & 5 & * \\ 0 & 0 & 0 & * \end{bmatrix}$

- If $a_{24}$ is a leading 1, then $a_{14}$ is zero, and we have our first possibility:

  $M_1 = \begin{bmatrix} 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $\quad \text{rank}(M_1) = 2 \quad \text{image}(M_1) = \text{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\} = \mathbb{R}^2$

  $\ker(M_1) = \text{span}\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -5 \\ 1 \\ 0 \end{bmatrix}\right\}$

- If $a_{24}$ is zero, then $a_{14}$ may be anything, say $n$. This is our second possibility:

  $M_2 = \begin{bmatrix} 0 & 1 & 5 & n \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $\quad \text{rank}(M_2) = 1 \quad \text{image}(M_2) = \text{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$

  $\ker(M_2) = \text{span}\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -5 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -n \\ 0 \\ 1 \end{bmatrix}\right\}$



(If $a_{24}$ is anything non-zero, then it must be a leading 1. Thus we have all possibilities.)

2. Consider the matrix

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$

a) [4 points] Construct an orthonormal eigenbasis for $A$.

$$\lambda I - A = \begin{bmatrix} \lambda-2 & -1 & -1 \\ -1 & \lambda-2 & -1 \\ -1 & -1 & \lambda-2 \end{bmatrix}$$

$$\begin{aligned} \det(\lambda I - A) &= (\lambda-2)^3 - 1 - 1 - 3(\lambda-2) \\ &= \lambda^3 - 6\lambda^2 + 9\lambda - 4 \\ &= (\lambda-1)(\lambda^2 - 5\lambda + 4) \\ &= (\lambda-1)^2(\lambda-4) \end{aligned}$$

A has eigenvalues 1 (alg. mult. 2) and 4 (alg. mult. 1).

$$\text{rref}(1\cdot I - A) = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad E_1 = \ker(1\cdot I - A) = \text{span}\left\{ \underbrace{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}}_{\vec{w}_1}, \underbrace{\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}}_{\vec{w}_2} \right\}$$

$$\text{rref}(4\cdot I - A) = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}, \quad E_4 = \ker(4\cdot I - A) = \text{span}\left\{ \underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}_{\vec{w}_3} \right\}$$

Then $\{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$ is an ~~orthonormal~~ eigenbasis for A, but it is not orthonormal.

Since A is symmetric, it will have an orthonormal eigenbasis. Let $\vec{v}_1 = \frac{\vec{w}_1}{\|\vec{w}_1\|} = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$.

Let $\vec{x}_2 = \vec{w}_2 - \text{Proj}_{\vec{v}_1}\vec{w}_2 = \vec{w}_2 - (\vec{w}_2 \cdot \vec{v}_1)\vec{v}_1 = \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$

Let $\vec{v}_2 = \frac{\vec{x}_2}{\|\vec{x}_2\|} = \frac{1}{\sqrt{6}}\begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$.

Then $\{\vec{v}_1, \vec{v}_2\}$ is an ONB for $E_1$.

Since A is symmetric, we know that $\vec{v}_1 \cdot \vec{w}_3 = 0$ and $\vec{v}_2 \cdot \vec{w}_3 = 0$.

Let $\vec{v}_3 = \frac{\vec{w}_3}{\|\vec{w}_3\|} = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Then $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is an orthonormal eigenbasis for A.

b) [2 points] Is $A$ diagonalizable? Why or why not?

Yes, since we have constructed an (orthonormal) eigenbasis. Let $S = [\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3]$.

Then $S^{-1}AS = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$.

(All symmetric matrices are diagonalizable.)

c) [4 points] Using parts (a) and (b), compute $A^{2000}$.

$$S = \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}.$$

$$B = S^{-1}AS = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$B^{2000} = (S^{-1}AS)^{2000} = S^{-1}A^{2000}S$$

$$A^{2000} = S B^{2000} S^{-1}$$

$$= S B^{2000} S^{T} \quad \text{(since } S \text{ is orthogonal)}$$

$$= \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2^{4000} \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

$$= \begin{bmatrix} & '' & \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}}2^{4000} & \frac{1}{\sqrt{3}}2^{4000} & \frac{1}{\sqrt{3}}2^{4000} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2}+\frac{1}{6}+\frac{1}{3}\cdot 2^{4000} & -\frac{1}{2}+\frac{1}{6}+\frac{1}{3}\cdot 2^{4000} & -\frac{2}{6}+\frac{1}{3}\cdot 2^{4000} \\ -\frac{1}{2}+\frac{1}{6}+\frac{1}{3}\cdot 2^{4000} & \frac{1}{2}+\frac{1}{6}+\frac{1}{3}\cdot 2^{4000} & -\frac{2}{6}+\frac{1}{3}\cdot 2^{4000} \\ -\frac{2}{6}+\frac{1}{3}\cdot 2^{4000} & -\frac{2}{6}+\frac{1}{3}\cdot 2^{4000} & \frac{4}{6}+\frac{1}{3}\cdot 2^{4000} \end{bmatrix}$$

3. Let $B$ be the matrix $\qquad \det B = \det(C)\cdot\det(F)$

$$B = \begin{bmatrix} 1.2 & -0.4 & 0 & 0 \\ 1.3 & 0.4 & 0 & 0 \\ 0 & 0 & 1.2 & 0.4 \\ 0 & 0 & -2.6 & 0.8 \end{bmatrix} = \left[\begin{array}{c|c} C & \\ \hline & F \end{array}\right]$$

a) [3 points] Find all eigenvalues of $B$.

$$\lambda I - B = \left[\begin{array}{cc|cc} \lambda - 1.2 & 0.4 & & \\ -1.3 & \lambda - 0.4 & & \\ \hline & & \lambda - 1.2 & -0.4 \\ & & 2.6 & \lambda - 0.8 \end{array}\right]$$

$$\begin{aligned} \det(\lambda I - B) &= \left[(\lambda-1.2)(\lambda-0.4) + (0.4)(1.3)\right]\left[(\lambda-1.2)(\lambda-0.8) + (0.4)(2.6)\right] \\ &= \left[\lambda^2 - 1.6\lambda + 0.48 + 0.52\right]\left[\lambda^2 - 2\lambda + 0.96 + 1.04\right] \\ &= (\lambda^2 - 1.6\lambda + 1)(\lambda^2 - 2\lambda + 2) \end{aligned}$$

eigenvalues : $\lambda_{1,2} = \dfrac{1.6 \pm \sqrt{2.56 - 4}}{2} = 0.8 \pm 0.6\,i$

$\lambda_{3,4} = \dfrac{2 \pm \sqrt{4 - 8}}{2} = 1 \pm i$

b) [3 points] Does the dynamical system

$$\vec{x}(t+1) = B\vec{x}(t)$$

have a point of stable equilibrium? Why or why not?

This is a discrete dynamical system.
Stable equilibria occur when the moduli of all eigenvalues are less than 1.

This is not the case here because

$|\lambda_{1,2}| = 1$ and $|\lambda_{3,4}| = \sqrt{2}$

c) [4 points] Describe qualitatively the behavior of this dynamical system if $\vec{x}(0)$ is the unit vector

$$\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

$\vec{x}(0) = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ may be written as a linear combination of the eigenvectors for $\lambda_1$ and $\lambda_2$ since its third and fourth components are zero.

Since $|\lambda_1| = 1$ and $|\lambda_2| = 1$, the trajectory of $\vec{x}(0)$ will be an ellipse.

4. Consider the linear transformation $T : C^\infty \to C^\infty$ given by

$$T(f) = f'' - 2f'.$$

a) [6 points] Find all real eigenvalues of $T$ and their corresponding eigenspaces.

Let $\lambda \in \mathbb{R}$ and suppose $\lambda$ is an eigenvalue for $T$.

Then $T(f) = \lambda f$ for some $f \in C^\infty$.

Suppose $f'' - 2f' = \lambda f$

$f'' - 2f' - \lambda f = 0$

This is equivalent to finding the kernel of the linear differential operator

$T - \lambda : C^\infty \to C^\infty$

$(T-\lambda) f = f'' - 2f' - \lambda f.$

The char. poly. of $T - \lambda$ is: $\quad P_{T-\lambda}(x) = x^2 - 2x - \lambda$

This has roots $\quad x = \dfrac{2 \pm \sqrt{4 + 4\lambda}}{2} = 1 \pm \sqrt{1+\lambda}$

These roots are distinct unless $\lambda = -1$.

When the roots of the char. poly. are distinct, we know

$$\ker(T-\lambda) = \text{span}\left\{ e^{(1+\sqrt{1+\lambda})t}, e^{(1-\sqrt{1+\lambda})t} \right\}$$

If $\lambda > -1$, then these two basis functions are real,

and we have $E_\lambda = \ker(T-\lambda) = \text{span}\left\{ e^{(1+\sqrt{1+\lambda})t}, e^{(1-\sqrt{1+\lambda})t} \right\}$.

If $\lambda < -1$, then we use Euler's formula:

$$e^{(1+i\sqrt{-1-\lambda})t} = e^t\left(\cos\sqrt{-1-\lambda}\,t + i \sin\sqrt{-1-\lambda}\,t\right)$$

$$e^{(1-i\sqrt{-1-\lambda})t} = e^t\left(\cos\sqrt{-1-\lambda}\,t - i \sin\sqrt{-1-\lambda}\,t\right)$$

and we have $E_\lambda = \ker(T-\lambda) = \text{span}\left\{ e^t \cdot \cos\sqrt{-1-\lambda}\,t,\ e^t \cdot \sin\sqrt{-1-\lambda}\,t \right\}$.

Special case: $\lambda = -1$.

$(T-1) f = f'' - 2f' + f$

$(T-1) f = 0$

when $f(t) = A t \cdot e^t + B \cdot e^t$, and $E_{-1} = \text{span}\{ t \cdot e^t, e^t \}$.

In conclusion, every $\lambda \in \mathbb{R}$ is an eigenvalue of $T$ with a two-dimensional eigenspace $E_\lambda$.

b) [2 points] Let $T_2$ be the same linear transformation restricted to the subspace $P_2$ of $C^\infty$, consisting of polynomials of degree at most 2. (That is, $T_2 : P_2 \to P_2$ is the transformation taking any polynomial $f$ of degree at most 2 to $f'' - 2f'$.)

Choose a basis for $P_2$, and write the matrix $A_2$ of $T_2$ with respect to this basis.

$P_2$ has a natural basis $\{1, x, x^2\}$.

$T_2(1) = 0$

$T_2(x) = -2$

$T_2(x^2) = 2 - 4x$

With respect to this basis, then, identifying $1 = \begin{bmatrix}1\\0\\0\end{bmatrix}$, $x = \begin{bmatrix}0\\1\\0\end{bmatrix}$, and $x^2 = \begin{bmatrix}0\\0\\1\end{bmatrix}$

$$A_2 = \begin{bmatrix} 0 & -2 & 2 \\ 0 & 0 & -4 \\ 0 & 0 & 0 \end{bmatrix}$$

c) [4 points] Find the image and kernel of this matrix $A_2$. Check (part of) your work by explaining the relationship between parts (a) and (c).

$$\text{rref}(A_2) = \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\ker(A_2) = \text{span}\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}\right\} = \text{span}\{1\}$$

↑ the first basis function

$$\text{image}(A_2) = \text{span}\left\{\begin{bmatrix}-2\\0\\0\end{bmatrix}, \begin{bmatrix}2\\-4\\0\end{bmatrix}\right\} = \text{span}\{-2, 2-4x\}$$

$$= \text{span}\{1, x\}$$

Any functions in $\ker(A_2) = \ker(T_2)$ should also be in $\ker(T)$, i.e. have eigenvalue 0.

We know from part (a), that $\ker(T) = E_0 = \text{span}\{e^{2t}, e^{0t}\}$

$= \text{span}\{e^{2t}, 1\}$

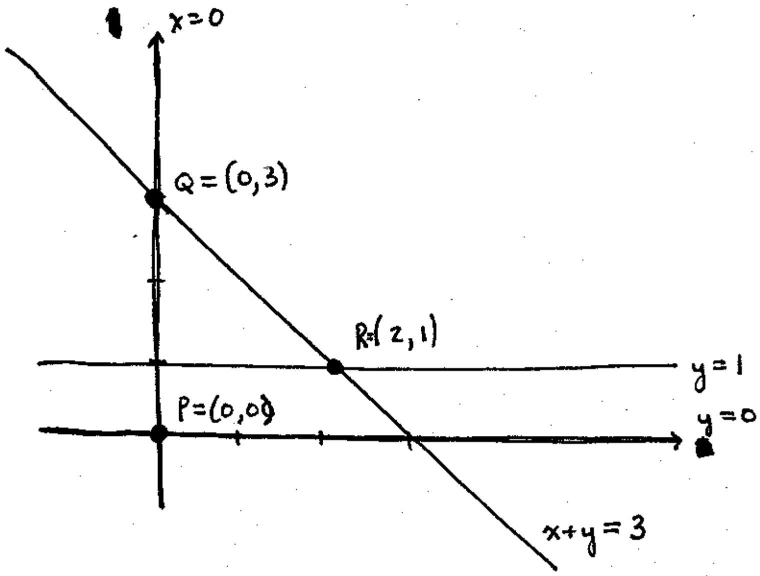
This checks for constant functions, but $e^{2t} \notin P_2$.

5. The following continuous dynamical system models the populations $x(t), y(t)$ of two species:

$$\frac{\partial x}{\partial t} = (y-1)x$$
$$\frac{\partial y}{\partial t} = (x+y-3)y$$

We consider only the first quadrant $x \geq 0, y \geq 0$ since we are modelling populations.

a) [2 points] Sketch the nullclines of this system, and find any equilibrium points.




b) [4 points] Use linear approximation to find the Jacobian $J$ of the system at any equilibrium points from part (a).

$$\begin{bmatrix} \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial t} \end{bmatrix} = \begin{bmatrix} f(x,y) \\ g(x,y) \end{bmatrix} = \begin{bmatrix} (y-1)x \\ (x+y-3)y \end{bmatrix}$$

$$\frac{\partial f}{\partial x} = y-1 \qquad \frac{\partial f}{\partial y} = x$$

$$\frac{\partial g}{\partial x} = y \qquad \frac{\partial g}{\partial y} = x+2y-3$$

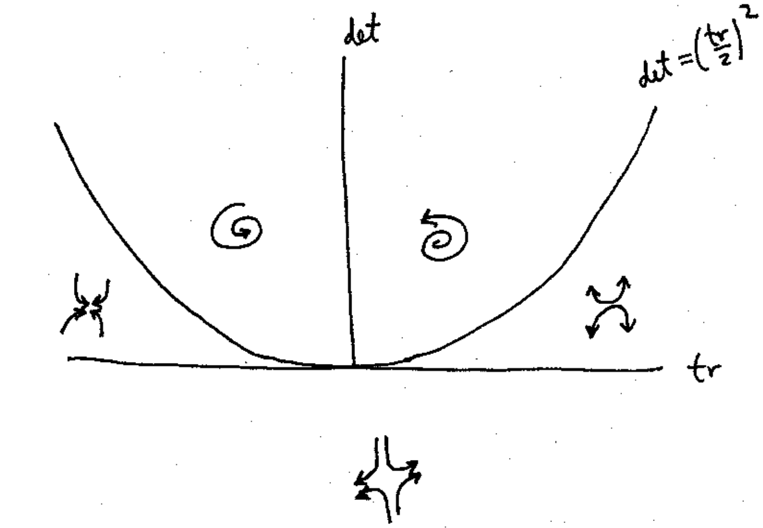
$$\begin{bmatrix} \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial t} \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{\partial f}{\partial x}(a,b) & \frac{\partial f}{\partial y}(a,b) \\ \frac{\partial g}{\partial x}(a,b) & \frac{\partial g}{\partial y}(a,b) \end{bmatrix}}_{J} \begin{bmatrix} x-a \\ y-b \end{bmatrix}$$

$$J(P) = \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix}$$

$$J(Q) = \begin{bmatrix} 2 & 0 \\ 3 & 3 \end{bmatrix}$$

$$J(R) = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}$$

Generally,




c) [4 points] Analyze the stability of each equilibrium point by computing the eigenvalues of $J$.

$J(P)$ has eigenvalues $-1, -3$ since it is diagonal.

Both are negative and real, so $P$ is a point of stable equilibrium

$\left( \det J(P) = 3 < \left(\frac{-4}{2}\right)^2, \right.$ so  $\left. \right)$

$J(Q)$ has eigenvalues $2, 3$ since it is lower triangular.

Both are positive and real, so $Q$ is not a point of stable equilibrium.

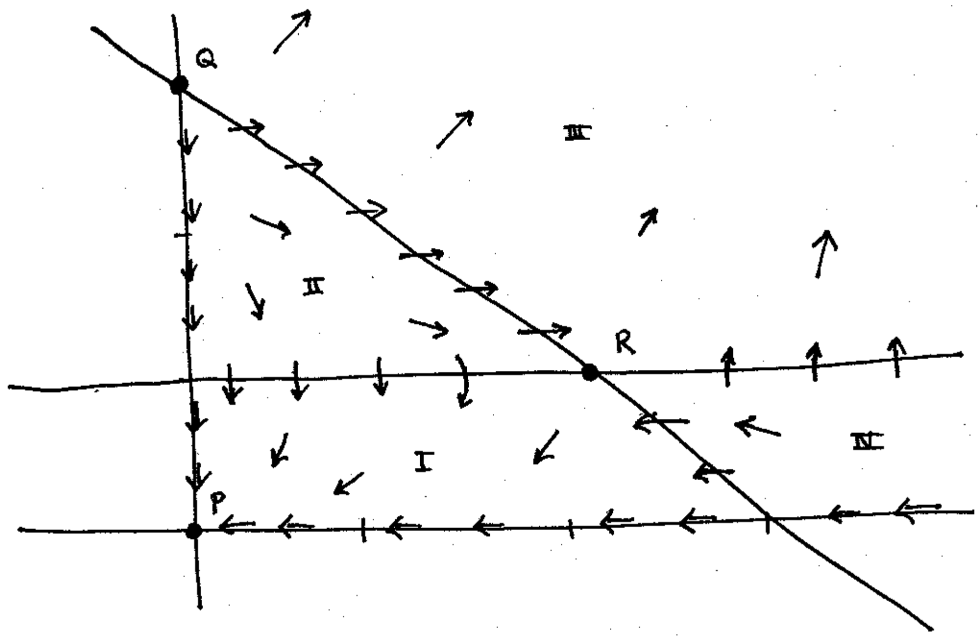
$\left( \det(Q) = 6 < \left(\frac{5}{2}\right)^2, \right.$ so  $\left. \right)$

$J(R)$ has char. poly. $f_{J(R)}(\lambda) = \lambda^2 - \lambda - 2$
$= (\lambda - 2)(\lambda + 1)$

and hence has eigenvalues $2, -1$.

With one positive and one negative, $R$ is not a point of stable eq.

d) [2 points] Sketch (approximately) the phase plane of this system, including behavior near equilibrium points and approximate direction of the flow lines in the regions separated by the nullclines.




| Region | Pt. | $\frac{\partial x}{\partial t}$ | $\frac{\partial y}{\partial t}$ | Flow |
|---|---|---|---|---|
| I | $(1, \frac{1}{2})$ | − | − | ↙ |
| II | $(\frac{1}{2}, 2)$ | + | − | ↘ |
| III | $(2,2)$ | + | + | ↗ |
| IV | $(3, \frac{1}{2})$ | − | + | ↖ |

Recall:

$$\frac{\partial x}{\partial t} = (y-1)x$$

$$\frac{\partial y}{\partial t} = (x+y-3)y$$

6. Consider the temperature $T(x,t)$ of a metal bar extending from $x = 0$ to $x = \pi$. The temperature satisfies the heat equation

$$\frac{\partial T}{\partial t} = \mu \frac{\partial^2 T}{\partial x^2}$$

and the ends are held at a constant temperature of zero, i.e. $T(0,t) = 0$ and $T(\pi,t) = 0$ for all $t \geq 0$.

a) [3 points] Show that the functions $e^{-\mu n^2 t}\sin(nx)$ satisfy the differential equation and the initial conditions for all positive integers $n$.

Let $T_n(x,t) = e^{-\mu n^2 t} \cdot \sin nx$.

Then $\frac{\partial T_n}{\partial t} = -\mu n^2 \cdot e^{-\mu n^2 t} \cdot \sin nx$

$\frac{\partial T_n}{\partial x} = n \cdot e^{-\mu n^2 t} \cdot \cos nx$

$\frac{\partial^2 T_n}{\partial x^2} = -n^2 \cdot e^{-\mu n^2 t} \cdot \sin nx$.

Hence $\frac{\partial T_n}{\partial t} - \mu \frac{\partial^2 T_n}{\partial x^2} = -\mu n^2 \cdot e^{-\mu n^2 t} \cdot \sin nx - \mu\left(-n^2 \cdot e^{-\mu n^2 t} \cdot \sin nx\right)$

$= 0$.

Also, $T_n(0,t) = e^{-\mu n^2 t} \cdot \sin 0 = 0$

$T_n(\pi,t) = e^{-\mu n^2 t} \cdot \sin n\pi = 0$ since $n \in \mathbb{N}$.

b) [7 points] Suppose now that the initial temperature of the bar is given by

$$T(x,0) = \theta(x) = \sin^3 x.$$

Determine $T(x,t)$ for all $x \in [0,\pi]$ and all times $t \geq 0$. [Hint: Use Euler's formula $e^{iy} = \cos y + i \sin y$ if you are not sure about the relevant trigonometric identities.] Examine the behavior of $T$ as $t \to \infty$.

We expect a solution of the type: $T(x,t) = \sum_{n=1}^{\infty} b_n \cdot e^{-\mu n^2 t} \cdot \sin nx$

The initial condition would read: $T(x,0) = \sum_{n=1}^{\infty} b_n \cdot \sin nx$

$$T(x,0) = \sin^3 x$$

So we hope that the first expression is the Fourier series of $\sin^3 x$.

We hope first for a finite linear combination of these functions that will satisfy the initial condition. (In fact, since $\sin^3 x$ is a nicely behaved periodic function composed of the basic trigonometric functions, its F. series is finite.)

Euler's Formula:

$$e^{3ix} = e^{i(3x)} = \cos 3x + i \sin 3x$$

$$e^{3ix} = (e^{ix})^3 = (\cos x + i \sin x)^3$$

$$= \cos^3 x + 3i \cos^2 x \sin x - 3 \cos x \sin^2 x - i \sin^3 x$$

$$= (\cos^3 x - 3 \cos x \cdot \sin^2 x) + i(3 \cos^2 x \sin x - \sin^3 x)$$

Matching imaginary parts yields:

$$\sin 3x = 3 \cos^2 x \cdot \sin x - \sin^3 x$$

$$= 3(1 - \sin^2 x) \sin x - \sin^3 x$$

$$= 3 \sin x - 4 \sin^3 x$$

and hence $\sin^3 x = \frac{3}{4} \sin x - \frac{1}{4} \sin 3x$ (This is the F. series for $\sin^3 x$ since F. series are unique!)

We let $b_1 = \frac{3}{4}$ and $b_3 = -\frac{1}{4}$.

Final solution: $T(x,t) = \frac{3}{4} e^{-\mu t} \cdot \sin x - \frac{1}{4} e^{-9\mu t} \cdot \sin 3x$