1 (a) FALSE

If A is invertible, $A\vec{x} = \vec{0} \Rightarrow A^{-1}(A\vec{x}) = A^{-1}\vec{0}$

$\Rightarrow \vec{x} = \vec{0}$. So there is only 1 solution.

(b) TRUE

Consider $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

$A^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, $A^4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $A^{17} = (A^4)^4 A = A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

(c) TRUE

Say $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$, at least one c_i nonzero

Then $A(c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3) = A(\vec{0}) = \vec{0}$

$= c_1 A\vec{v}_1 + c_2 A\vec{v}_2 + c_3 A\vec{v}_3 = \vec{0}$

(d) TRUE

Any transformation from $\mathbb{R}^2$ to $\mathbb{R}$ can be

represented as $[a \ b] = [1 \ 0]\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$

(e) FALSE

Consider a reflection about the x-axis.

(f) FALSE

If $c = -1$, the second col is the sum of the first and third columns.

(g) TRUE

Reasoning same as in solution to (a)

(h) TRUE

Consider $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Anything in the form $A - I_2$, where A is a shear, will work.

(i) FALSE

Consider $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. $BA = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

$\ker BA = \operatorname{Span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$ $\ker B = \operatorname{Span}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$

(j) TRUE

If $\vec{v} \in \ker A$, $\vec{v} \neq \vec{0}$, then $\vec{v} \in \operatorname{Im}(A - I_n)$ because $A\vec{v} = \vec{0} \Rightarrow A(-\vec{v}) = \vec{0} \Rightarrow (A - I_n)(-\vec{v}) = \vec{v}$

So $\ker A \subseteq \operatorname{Im}(A - I_n)$
and $\dim \ker A \leq \operatorname{rank}(A - I_n)$

② Gauss-Jordan Elimination: Assume $1-k^2 \neq 0$

$$\left[\begin{array}{ccc|c} 1 & k & 1 & 1 \\ 1 & k & -1 & 3 \\ k & 1 & 0 & 2 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & k & 1 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 1-k^2 & -k & 2-k \end{array}\right] \begin{array}{l} \\ \frac{1}{2}(\mathrm{II}-\mathrm{I}) \\ \mathrm{III}-k\,\mathrm{I} \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & k & 1 & 1 \\ 0 & 1-k^2 & -k & 2-k \\ 0 & 0 & 1 & -1 \end{array}\right] \begin{array}{l} \\ \mathrm{III} \\ -\mathrm{II} \end{array} \rightarrow \left[\begin{array}{ccc|c} 1 & k & 1 & 1 \\ 0 & 1 & \frac{k}{k^2-1} & \frac{k-2}{k^2-1} \\ 0 & 0 & 1 & -1 \end{array}\right] \frac{\mathrm{II}}{1-k^2}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1-\frac{k^2}{k^2-1} & 1-\frac{k^2-2k}{k^2-1} \\ 0 & 1 & \frac{k}{k^2-1} & \frac{k-2}{k^2-1} \\ 0 & 0 & 1 & -1 \end{array}\right] \mathrm{I}-k\,\mathrm{II} \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{1-k^2} & \frac{2k-1}{k^2-1} \\ 0 & 1 & \frac{k}{k^2-1} & \frac{k-2}{k^2-1} \\ 0 & 0 & 1 & -1 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{2k-2}{k^2-1} \\ 0 & 1 & 0 & \frac{2k-2}{k^2-1} \\ 0 & 0 & 1 & -1 \end{array}\right]$$

$$x = \frac{2k-2}{k^2-1} = \frac{2}{k+1}$$

$$y = \frac{2k-2}{k^2-1} = \frac{2}{k+1}$$

$$z = -1$$

If $k=1$

$$x+y+z = 1$$
$$x+y-z = 3$$
$$x+y = 2$$

Row-reduce

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} + \begin{bmatrix} 2-a \\ a \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} + a \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

If $k=-1$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 \end{array}\right]$$

INCONSISTENT

No solutions.

(3) (a) Let T be the reflection about $L = \text{Span}\left(\begin{bmatrix} -1 \\ 2 \end{bmatrix}\right)$

$$T\vec{v} = 2\,\text{proj}_L \vec{v} - \vec{v}$$

$\text{proj}_L \vec{v} = (\vec{v} \cdot \vec{u})\,\vec{u}$, where L is spanned by unit vector $\vec{u}$

$$T\vec{e}_1 = 2\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}\right)\begin{bmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$$

$$= -\frac{2}{\sqrt{5}}\begin{bmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix} = \begin{bmatrix} 2/5 \\ -4/5 \end{bmatrix}$$

$$T\vec{e}_2 = 2\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}\right)\begin{bmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$$

$$= \frac{4}{\sqrt{5}}\begin{bmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix} = \begin{bmatrix} -4/5 \\ 8/5 \end{bmatrix}$$

So the matrix for T is $\begin{bmatrix} | & | \\ T\vec{e}_1 & T\vec{e}_2 \\ | & | \end{bmatrix}$

$$= \begin{bmatrix} 2/5 & -4/5 \\ -4/5 & 8/5 \end{bmatrix}$$

(b) If T is a shear parallel to L,

$$T\vec{v} - \vec{v} = \vec{0} \quad \text{for all } \vec{v} \in L$$

That is,

$$\begin{bmatrix} 7 & -4 \\ 9 & -5 \end{bmatrix} \vec{v} - \vec{v} = \vec{0}$$

$$\left(\begin{bmatrix} 7 & -4 \\ 9 & -5 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \vec{v} = \vec{0}$$

$$\begin{bmatrix} 6 & -4 \\ 9 & -6 \end{bmatrix} \vec{v} = \vec{0}$$

$$\text{Ker} \begin{bmatrix} 6 & -4 \\ 9 & -6 \end{bmatrix} = \text{Span}\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix} \right)$$

So $L = \text{span}\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix} \right)$

④ (a) NO

Assume BA has rank 3. Then both A and B have rank 3.

- If A doesn't have rank 3, A has a nontrivial kernel, so there's a nonzero vector $\vec{v}$ s.t. $A\vec{v} = \vec{0}$, which means $BA\vec{v} = \vec{0}$, which would mean BA doesn't have rank 3

- If A has rank 3 but B doesn't, there's a nonzero vector $\vec{y}$ s.t. $B\vec{y} = \vec{0}$ and a vector $\vec{x}$ s.t. $A\vec{x} = \vec{y}$. So $BA\vec{x} = B\vec{y} = \vec{0}$, which means BA doesn't have rank 3.

But if rank (A) = 3, rank (B) = 3, A and B are invertible, so AB is invertible. So rank (AB) = 3.

Contradiction – so rank (BA) ≠ 3

(b) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

$AB = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $BA = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ rank (BA) = 2
rank (AB) = 1

(c) $A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $AB = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $BA = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

rank (AB) = 1 rank (BA) = 0