Last Name:____________________________

First Name:____________________________

Mathematics 21b

Second Midterm
April 26, 1999

Your Section (circle one):

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MWF9	MWF 10	MWF 10	MWF 11	MWF 11	MWF 12	TuTh 10	TuTh 11:30

Question	Points	Score
1	18	
2	18	
3	16	
4	12	
5	20	
6	16	
Total	100	

No calculators are allowed.
Justify your answers carefully (except for question 1).
Except for question 1, no credit can be given for unsubstantiated answers.

1. For each of the following, circle T for true or F for false. No explanation is necessary.

(a) T F If S, T are $n \times n$ matrices such that $ST = TS$, and $\vec{v}$ is an eigenvector of S, then $T\vec{v}$ is also an eigenvector of S.

(b) T F If $3 + i$ is an eigenvalue of a real 3 matrix A, then A is diagonalizable over $\mathbf{C}$.

(c) T F If A is a real 5×4 matrix, then AA^T is positive definite.

(d) T F If $\vec{v_1}, \vec{v_2}, \vec{v_3}, \vec{v_4}$ are row vectors, then

$$\det \begin{bmatrix} -\vec{v_1}- \\ -\vec{v_2}- \\ -\vec{v_3}- \\ -\vec{v_4}- \end{bmatrix} = \det \begin{bmatrix} -\vec{v_2}- \\ -\vec{v_4}- \\ -\vec{v_3}- \\ -\vec{v_1}- \end{bmatrix}.$$

(e) T F A 2×2 real matrix A with $\det(A) < 0$ has 2 distinct real eigenvalues.

(f) T F If $\operatorname{tr}(A) > 0$ then in the dynamical system $\frac{d}{dt}\vec{x} = A\vec{x}(t)$, $\vec{0}$ is not asymptotically stable.

2. a) Let

$$\vec{u} = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 3 \end{bmatrix}, \qquad \vec{v} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \end{bmatrix}$$

Calculate the area of the parallelogram formed by $\vec{u}$ and $\vec{v}$ (i.e. two of its sides are equal to $\vec{u}$ and $\vec{v}$).

b) Is the matrix

$$A = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 & 8 \\ 0 & 0 & 9 & 10 & 11 \\ 0 & 0 & 0 & 12 & 13 \\ 0 & 0 & 0 & 0 & 14 \end{bmatrix}$$

diagonalizable over $\mathbf{R}$? Briefly explain why or why not.

c) Find a matrix with eigenvalues equal to 2,3,5,7.

d) Which of the following matrices A, B, C, D, E has characteristic polynomial

i) $\lambda^4 - 8\lambda^3 - 26\lambda^2 - 88\lambda + 121$?

Answer:____________

ii) $\lambda^4 - 8\lambda^3 - 19\lambda^2 + 98\lambda - 72$?

Answer:____________

For each polynomial there is exactly one correct answer.

$$A = \begin{bmatrix} 6 & 1 & 3 & 1 \\ 2 & -2 & 0 & -2 \\ -1 & 4 & 1 & 4 \\ 2 & 3 & 3 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} -5 & 1 & 2 & 6 \\ 8 & -4 & 4 & 1 \\ -2 & 5 & 4 & 0 \\ 3 & -7 & 6 & -3 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & -11 & 0 & 0 \\ 1 & -4 & 2 & 1 \\ 0 & 0 & 11 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$D = \begin{bmatrix} 11 & 0 & 0 & 0 \\ -6 & -1 & 0 & 0 \\ 8 & 2 & 11 & 0 \\ 1 & 3 & 4 & -1 \end{bmatrix}, \quad E = \begin{bmatrix} 9 & 4 & 5 & 9 \\ 0 & 2 & 2 & 7 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & -4 \end{bmatrix}$$

3. A is a real $n \times n$ matrix such that $A^2 = -I_n$.

a) Show that A is invertible.

b) Show that n must be even.

c) Show that A has no real eigenvalues.

4. Consider a linear transformation $T : \mathbf{R}^2 \to \mathbf{R}^2$. Suppose the matrix of T with respect to the basis $\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ is $\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$. Find the matrix of T with respect to the basis $\begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \end{bmatrix}$.

5. Consider the quadratic form

$$q(x_1, x_2, x_3) = 2x_1^2 + 2x_2^2 + 2x_3^2 - 2x_1x_2 - 2x_2x_3 + 2x_3x_1$$

a) Find a symmetric matrix A such that

$$q(\vec{x}) = \vec{x}^T A\vec{x}$$

for all $\vec{x}$ in $\mathbf{R}^3$.

b) Find all the eigenvalues of A and their algebraic and geometric multiplicities.

c) Is A positive definite? Briefly justify your answer.

d) Find an orthonormal eigenbasis for A.

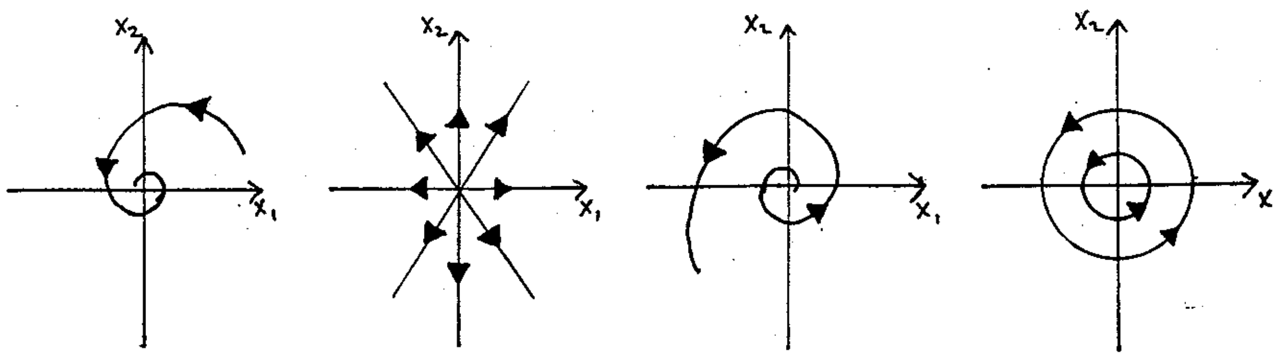
6. Consider the dynamical system

$$\frac{dx_1}{dt} = ax_1 - bx_2$$

$$\frac{dx_2}{dt} = bx_1 + ax_2$$

a) Sketch a phase portrait in the case $a = 1$, $b = -1$.

b) For each of the four phase portaits below, indicate (by circling the correct answer) whether the constants a and b are positive, negative, or zero.



a	>0	b	>0	a	>0	b	>0	a	>0	b	>0	a	>0	b	>0
	$=0$		$=0$		$=0$		$=0$		$=0$		$=0$		$=0$		$=0$
	<0		<0		<0		<0		<0		<0		<0		<0