

Previous Second Midterms in Math 21b

Math 21b Exam 2 - April 10, 1995

(All questions worth 20%)

1) Find the determinant of $\mathbf{A} = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & 1 \\ 3 & 2 & 1 & 0 \end{bmatrix}$.

2) Find the least squares solution of $\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$.

3) Let $\mathbf{A} = \begin{bmatrix} 0 & 0 & 3 \\ 0 & 2 & 0 \\ 3 & 0 & 0 \end{bmatrix}$.

- Find the characteristic polynomial $f_{\mathbf{A}}(\lambda)$ of \mathbf{A} .
- Find the eigenvalues and corresponding eigenvectors of \mathbf{A} .

c) Find $\mathbf{A}^{100} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

4) Let $\mathbf{M} = \begin{bmatrix} 2 & -3 \\ 1 & -1 \end{bmatrix}$.

- Find the complex eigenvalues of \mathbf{A} . Write your answers in polar form.
- Determine the (asymptotic) stability of the dynamical system $\vec{x}(t+1) = \mathbf{M}\vec{x}(t)$. Describe the long term behavior of the system qualitatively.
- Is there a positive k such that $\mathbf{M}^k = I_2$? If so, find the smallest such k . If not, explain why not.

5) Consider three vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ in \mathbf{R}^3 . In the table

below, we list all the dot products $\vec{v}_i \cdot \vec{v}_j$.

\cdot	\vec{v}_1	\vec{v}_2	\vec{v}_3
\vec{v}_1	4	0	0
\vec{v}_2	0	2	1
\vec{v}_3	0	1	5

Find the volume of the box (= parallelepiped) formed by \vec{v}_1, \vec{v}_2 , and \vec{v}_3 .

Math 21b Exam 2 - Dec 12, 1995

(All questions worth 20%)

Problem 1 is missing.

2) a) Find the determinant of the matrix

$$\mathbf{A} = \begin{bmatrix} 2 & 0 & 0 & -1 \\ 0 & 2 & 0 & -1 \\ 0 & 0 & 2 & -1 \\ -1 & -1 & -1 & 2 \end{bmatrix}$$

b) For a positive arbitrary integer n , find the determinant of the matrix $\begin{bmatrix} \mathbf{I}_n & \mathbf{I}_n \\ \mathbf{I}_n & \mathbf{0} \end{bmatrix}$. (The $\mathbf{0}$ stands for the $n \times n$ matrix whose entries are all zero.) Your answer may depend on n . Explain your reasoning carefully.

3) Solve the initial value problem:

$$\frac{d\vec{x}}{dt} = \begin{bmatrix} 5 & -3 \\ 9 & -7 \end{bmatrix} \vec{x}, \quad \vec{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Sketch the trajectory, clearly indicating the long term behavior in the future and the past.

4) Find a real 2×2 matrix \mathbf{A} such that $\vec{x}(t) = \begin{bmatrix} 2^t \\ 1 - 2^t \end{bmatrix}$ is a trajectory of the dynamical system $\vec{x}(t+1) = \mathbf{A}\vec{x}(t)$.

5) Consider the quadratic form

$$q(x_1, x_2, x_3) = 6x_1^2 + 6x_2^2 + 6x_3^2 + 8x_1x_2 + 6x_2x_3.$$

- Find a symmetric matrix \mathbf{A} such that $q(\vec{x}) = \vec{x}^T \mathbf{A} \vec{x}$ for all vectors \vec{x} in \mathbf{R}^3 .
- Compute the eigenvalues of the matrix \mathbf{A} you just found in part (a).
- Determine the definiteness of the quadratic form q .
- Find the minimal value the quadratic form takes on the unit sphere, i.e. among the vectors \vec{x} in \mathbf{R}^3 with $x_1^2 + x_2^2 + x_3^2 = 1$.

Math 21b Exam 2 - April 23, 1996

1) Let $L: \mathbf{R}^3 \rightarrow \mathbf{R}^3$ be a reflection in the line spanned by

$$\vec{v} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}.$$

- Find an orthonormal eigenbasis B for L .
- Find \mathbf{B} , the matrix representation for L with respect to the eigenbasis B .
- Find \mathbf{A} , the matrix representation for L with respect to the standard basis.

2) Let \vec{x} be the vector $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ in \mathbf{R}^4 ,

and let W be the subspace $\text{Span} \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$ of \mathbf{R}^4 .

Find the angle between \vec{x} and its projection onto W .

3) Consider the initial value problem

$$\vec{x}(t+1) = \begin{bmatrix} 5 & -6 \\ 3 & -1 \end{bmatrix} \vec{x}(t), \quad \vec{x}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Find closed formulas for the components of $\vec{x}(t)$.

Angles which appear in your answers do not need to be computed numerically.

4) Is there a real 3×3 matrix \mathbf{A} with all of the following properties:

- All eigenvalues of \mathbf{A} over \mathbb{C} (the complex numbers) are integers
- \mathbf{A} is not diagonalizable over \mathbb{C}
- $\det(\mathbf{A}^T \mathbf{A}) = 36$?

Give an example or show none can exist.

5) Consider the matrix $\mathbf{A} = \begin{bmatrix} p & -q & -r & -s \\ q & p & s & -r \\ r & -s & p & q \\ s & r & -q & p \end{bmatrix}$ where

$p, q, r,$ and s are arbitrary real numbers. Note that the columns of \mathbf{A} are orthogonal.

- Compute $\mathbf{A}^T \mathbf{A}$.
- Find the singular values of \mathbf{A} . (not relevant, Spr 99)
- Find the absolute value of the determinant of \mathbf{A} .
- For which choices of $p, q, r,$ and s is \mathbf{A} invertible?
- Find the eigenvalues of \mathbf{A} in \mathbb{C} .
- If \vec{x} is a vector in \mathbf{R}^4 , what is the relationship between $\|\vec{x}\|$ and $\|\mathbf{A} \vec{x}\|$?
- Consider the numbers $65 = 2^2 + 3^2 + 4^2 + 6^2$ and $84 = 1^2 + 3^2 + 5^2 + 7^2$. Show that the product $5460 = 65 \cdot 84$ can be expressed as the sum of the squares of four integers: $5460 = a^2 + b^2 + c^2 + d^2$. You need not find the numbers $a, b, c,$ and d . (Hint: Part (f) is useful.)

Math 21b Exam 2 - Tuesday, November 26, 1996

- Find the quadratic function $f(x) = a + bx + cx^2$ that best fits the points $(-1,-2), (-1,-1), (0,0), (1,-1), (1,-2)$ in the sense of least squares.
 - Does the graph of the function you found in part a) pass through the origin?

2a) Calculate the determinant of the matrix

$$\mathbf{A} = \begin{bmatrix} 2 & 0 & 1 & 0 & 3 \\ 1 & 2 & 1 & 0 & 0 \\ -4 & 0 & 2 & -1 & 0 \\ 3 & 0 & -3 & 0 & 0 \\ -3 & 0 & 1 & 0 & 0 \end{bmatrix}$$

2b) Consider the 8×8 matrix

$$\mathbf{A} = \begin{bmatrix} a & b & b & b & b & b & b & b \\ a & a & b & b & b & b & b & b \\ a & a & a & b & b & b & b & b \\ a & a & a & a & b & b & b & b \\ a & a & a & a & a & b & b & b \\ a & a & a & a & a & a & b & b \\ a & a & a & a & a & a & a & b \\ a & a & a & a & a & a & a & a \end{bmatrix}.$$

(The i -th row has i a 's, followed by $(8-i)$ b 's.)

- Find $\det(\mathbf{A})$.
- If $a = 1$ and $b = -1$, is \mathbf{A} invertible?

3) We are given three vectors in \mathbf{R}^4 :

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ -2 \\ 1 \\ -1 \end{bmatrix}, \quad \text{and} \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ -1 \\ 3 \\ -1 \end{bmatrix}.$$

- Find the length of \mathbf{v}_1 .
- Find the area of the parallelogram determined by the vectors $\{\mathbf{v}_1, \mathbf{v}_2\}$.
- Using the vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ above, find an orthonormal basis $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ for the subspace which they span, using the Gram-Schmidt orthogonalization method.
- Given the matrix $\mathbf{B} = [\mathbf{w}_1 \ \mathbf{w}_2]$ where \mathbf{w}_1 and \mathbf{w}_2 are as found in the previous problem. Find the eigenvalues of the matrix $\mathbf{B}\mathbf{B}^T$, and their algebraic and geometric multiplicities. Describe the eigenspaces. (Hint: Do no further calculation! There is no need to find the characteristic polynomial.)

4) Consider the dynamical system

$$\mathbf{x}(t+1) = \mathbf{A} \mathbf{x}(t), \quad \text{with} \quad \mathbf{A} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

- Find the eigenvalues of \mathbf{A} .
- If $\mathbf{x}(0) = \begin{bmatrix} -0.1 \\ 0.2 \end{bmatrix}$, will the trajectory of $\mathbf{x}(t)$ enter all four quadrants?
- What is $\mathbf{x}(12)$?

5) Consider a dynamical system $\mathbf{x}(t+1) = \mathbf{A} \mathbf{x}(t)$, where \mathbf{A} is a 3×3 matrix. Suppose that $\mathbf{x}(0), \mathbf{x}(1)$, and $\mathbf{x}(2)$ are linearly independent. Further suppose that $\mathbf{x}(3) = a\mathbf{x}(0) + b\mathbf{x}(1)$.

- Find the matrix of \mathbf{A} relative to the basis $\mathbf{B} = \{\mathbf{x}(0), \mathbf{x}(1), \mathbf{x}(2)\}$, i.e. find $[\mathbf{A}]_{\mathbf{B}}$.

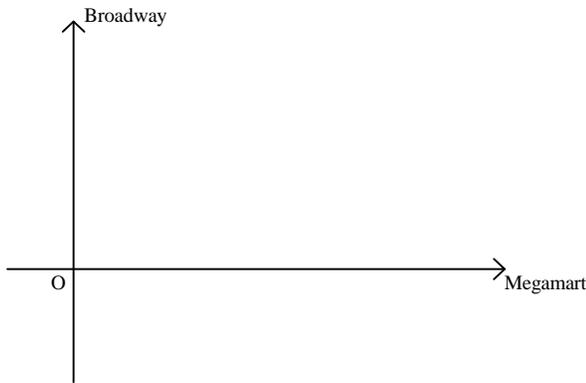
b) If we are further given that $\mathbf{x}(0) = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{2} \end{bmatrix}$, $\mathbf{x}(1) = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$,

$\mathbf{x}(2) = \begin{bmatrix} 0 \\ -5 \\ 0 \end{bmatrix}$, and $\mathbf{x}(3) = \begin{bmatrix} 9 \\ 0 \\ 1 \end{bmatrix}$.

Determine the matrix \mathbf{A} relative to the standard basis of \mathbf{R}^3 .

6) Not long ago, the Broadway Marketplace opened a couple of blocks from here, selling fresh meat and produce and some groceries. For argument's sake, let's say that this store has one competitor, Megamart Inc., and that there is a constant total customer base of 5,000 people, all of whom originally went to Megamart. Weekly surveys show that Megamart retains 80% of its customers from the previous week with the rest going to Broadway. Broadway keeps 70% of its customers from one week to the next, with the rest going for the cheaper prices at the Megamart.

a) Indicate, on the axes provided, the trajectory associated with this discrete dynamical system.



b) Will there come a point where the number of people going to each store will remain about the same from week to week? If so, how many will be going to each store? For full credit, you must justify all of your conclusions.

Spring 98 Exam 2

1) Let \mathbf{B} be the matrix $\mathbf{B} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$.

- Find $\det(\mathbf{B})$.
- Find all eigenvalues (real and/or complex) of \mathbf{B} and their **algebraic** multiplicities.
- For each **real** eigenvalue of \mathbf{B} , find the eigenspace corresponding to it and its geometric multiplicity. (Clearly indicate which eigenspaces and multiplicities belong to which eigenvalues.)

2) For each of the following, circle T for true or F for false. No explanation is necessary. Assume all the matrices \mathbf{A} and \mathbf{B} below are arbitrary $n \times n$ matrices with real entries.

- T F** It is always the case that $\det(\mathbf{A}^T \mathbf{A}) \geq 0$.
- T F** If \mathbf{A} and \mathbf{B} are symmetric, the \mathbf{AB} is symmetric.
- T F** If \vec{v} is an eigenvector of \mathbf{B} , then \vec{v} is an eigenvector of \mathbf{AB} .
- T F** A linear transformation $T: \mathbf{R}^n \rightarrow \mathbf{R}^n$ is orthogonal if and only if $T(\vec{x}) \cdot T(\vec{y}) = \vec{x} \cdot \vec{y}$ for all vectors \vec{x} and \vec{y} in \mathbf{R}^n .
- T F** If \mathbf{C} is any $m \times n$ matrix and \vec{b} is any vector in \mathbf{R}^m , then the equation $\mathbf{C}\vec{x} = \vec{b}$ always has a unique least squares solution.

3) Find the determinant of the matrix

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ -1 & 0 & 3 & 4 & 5 & 6 & 7 & 8 \\ -1 & -2 & 0 & 4 & 5 & 6 & 7 & 8 \\ -1 & -2 & -3 & 0 & 5 & 6 & 7 & 8 \\ -1 & -2 & -3 & -4 & 0 & 6 & 7 & 8 \\ -1 & -2 & -3 & -4 & -5 & 0 & 7 & 8 \\ -1 & -2 & -3 & -4 & -5 & -6 & 0 & 8 \\ -1 & -2 & -3 & -4 & -5 & -6 & -7 & 0 \end{bmatrix}$$

4) a) Consider the matrix $\mathbf{A} = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 1 & -1 \\ 4 & 0 & 8 \\ -1 & 1 & 1 \\ -3 & 1 & -1 \end{bmatrix}$.

Use the Gram-Schmidt process to find an orthonormal basis for the image of \mathbf{A} .

- Find an orthogonal matrix \mathbf{Q} and an upper triangular matrix \mathbf{R} such that $\mathbf{A} = \mathbf{QR}$.
- Show that for an arbitrary matrix \mathbf{A} (not just the particular matrix \mathbf{A} used in parts a and b), the least squares solution of $\mathbf{A}\vec{x} = \vec{b}$ simplifies to $\vec{x} = \mathbf{R}^{-1} \mathbf{Q}^T \vec{b}$ (assuming that $\mathbf{A}^T \mathbf{A}$ is invertible). Here $\mathbf{A} = \mathbf{QR}$ is a decomposition of the arbitrary matrix \mathbf{A} into a product of an orthogonal matrix \mathbf{Q} and an upper triangular matrix \mathbf{R} .

5) Suppose 2500 TV watchers all watch exactly one rerun each night: *Star Trek* or *Melrose Place*. Assume 70% of those watching *Star Trek* on a given night watch it again the next night, and 80% of the *Melrose Place* viewers watch *Melrose Place* the next night. On April 8, 1998 there are 2024 *Star Trek* viewers.

- How many viewers watch *Star Trek* on April 18, 1998?
- It's a summer night early in the next century. How many viewers watch each show that night? (Round your answers to the nearest integer.)