Solution Set for Mathematics 21b Final Exam, given January 26, 1993

True or False section:

- If 0 is an eigenvalue of the matrix A, then $\det(A) = 0$.

  True. The determinant of a matrix is the product of its eigenvalues.

- The set of solutions of $\frac{d^2x}{dt^2} + \frac{dx}{dt} - 2x = 0$ such that $\lim_{t \to \infty} x(t) = 0$ forms a vector space of dimension 1.

  True. The general solution is $x(t) = c_1 e^{t} + c_2 e^{-2t}$; solutions of the form $c_2 e^{-2t}$ satisfy the requirements.

- If A is a 4×4 matrix, then $\det(-A) = -\det(A)$.

  False. If A is $n \times n$, $\det(-A) = (-1)^n \det(A)$.

- The matrix
$$\begin{bmatrix} \cos\theta & -\sin\theta & 1 \\ \sin\theta & \cos\theta & 2 \\ 0 & 0 & 3 \end{bmatrix}$$
is invertible for all θ.

  True. The determinant $= 3 \neq 0$

- The vectors
$$\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$
are linearly independent.

  False. If we call these vectors $\vec{v}_1, \ldots, \vec{v}_4$, then there is a nontrivial linear relation $\vec{v}_1 - \vec{v}_2 + \vec{v}_3 - \vec{v}_4 = 0$.

- The curve $x^2 - 6xy + y^2 = 4$ is an ellipse.

  False. The corresponding symmetric matrix for this quadratic form is $\begin{bmatrix} 1 & -3 \\ -3 & 1 \end{bmatrix}$ with eigenvalues $4, -2$ of opposite sign: a hyperbola.

1. Consider the matrix

$$A = \begin{bmatrix} 0 & 2 & 0 & 0 \\ k & 0 & 2 & 0 \\ 0 & k & 0 & 2 \\ 0 & 0 & k & 0 \end{bmatrix}$$

where k is a constant.

a. Find $\det(A)$: Using the cofactor expansion,

$$\det A = -k \det \begin{bmatrix} 2 & 0 & 0 \\ k & 0 & 2 \\ 0 & k & 0 \end{bmatrix} = -2k \det \begin{bmatrix} 0 & 2 \\ k & 0 \end{bmatrix} = \underline{\underline{4k^2}}$$

b. For which choice of k is A non-invertible?
For $\underline{\underline{k=0}}$.

c. For this case ($k=0$) find a basis for the kernel and range of A:
In this case, A is in reduced row-echelon form, and the solution is by inspection:

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \underline{\text{basis for } \ker(A)} \qquad \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\} \quad \underline{\text{basis for range}(A)}$$

2. Consider the matrix A given in problem 1.

a. Find a value of k for which A is diagonalizable:
A quick solution is to realize that if $\underline{k=2}$ then A is a symmetric matrix, and therefore is diagonalizable.

b. Find a value for k for which A is not diagonalizable:

80.2

A quick solution here is to realize that with $k=0$, A has only the eigenvalue $\lambda=0$ (with multiplicity four), and only one eigenvector, namely the basis vector for $\ker(A)$ found in problem 1. In this case, A has no eigenbasis and is not diagonalizable.

3 Find the inverse of the matrix

$$A = \begin{bmatrix} 0 & 2 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Using row-reduction (Gaussian elimination):

$$\left[\begin{array}{cccc|cccc} 0 & 2 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 & 1 & 0 & 0 & 0 \end{array}\right] \rightarrow$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -4 & 1 & 0 & -2 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -\frac{1}{4} & 0 & \frac{1}{2} & 0 \end{array}\right] \rightarrow$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 1 & 0 & -2 \\ 0 & 1 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -\frac{1}{4} & 0 & \frac{1}{2} & 0 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 0 & 1 & 0 & -2 \\ \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{1}{4} & 0 & \frac{1}{2} & 0 \end{bmatrix}$$

(4) Find the solution of the differential equation

$$\frac{d^2x}{dt^2} - 3\frac{dx}{dt} + 2x = 0 \; ; \quad x(0) = 0 \, , \; x(\ln 2) = 1 .$$

Solutions are of the form $e^{\lambda t}$, where λ satisfies the eigenvalue equation

$$\lambda^2 - 3\lambda + 2 = 0$$

$$\lambda = \frac{1}{2}\left(3 \pm \sqrt{9 - 4 \cdot 2}\right) = \frac{1}{2}(3 \pm 1) = 1, 2$$

General solution is $x(t) = c_1 e^{t} + c_2 e^{2t}$. To incorporate the conditions $x(0) = 0$, $x(\ln 2) = 1$:

$$x(0) = c_1 + c_2 = 0 \implies c_2 = -c_1$$
$$x(\ln 2) = c_1 e^{\ln 2} + c_2 e^{2\ln 2} = 2c_1 + 4c_2 = 1$$
$$\implies 2c_1 - 4c_1 = -2c_1 = 1 \implies c_1 = -\tfrac{1}{2}$$

So

$$\underline{\underline{x(t) = -\tfrac{1}{2} e^{t} + \tfrac{1}{2} e^{2t}}}$$ unique solution.

(5) A rabbit population and a wolf population are modelled by the equations

$$r(t+1) = 5r(t) - 2w(t)$$
$$w(t+1) = r(t) + 2w(t)$$

with initial populations $r(0) = 300$, $w(0) = 200$

(a) Find formulas for $r(t)$ and $w(t)$:

$$\vec{x}(t) = \begin{bmatrix} r(t) \\ w(t) \end{bmatrix} \quad \vec{x}(t+1) = A\,\vec{x}(t) \quad A = \begin{bmatrix} 5 & -2 \\ 1 & 2 \end{bmatrix}$$

The solution will be given by $\vec{x}(t) = c_1 \lambda_1^t \vec{v}_1 + c_2 \lambda_2^t \vec{v}_2$ where c_1, c_2 are constants to be determined by the initial conditions, λ_1, λ_2 are the eigenvalues of A, $\vec{v}_1, \vec{v}_2$ are the eigenvectors of A. To find λ_1, λ_2:

$$\det(A-\lambda I) = \det\begin{bmatrix} 5-\lambda & -2 \\ 1 & 2-\lambda \end{bmatrix} = (5-\lambda)(2-\lambda)+2$$

$$= \lambda^2 - 7\lambda + 12 = 0$$

$$\lambda = \tfrac{1}{2}\left(7 \pm \sqrt{49-4\cdot 12}\right) = \tfrac{1}{2}(7 \pm 1) \Longrightarrow \lambda_1 = 3,\ \lambda_2 = 4$$

To determine the corresponding eigenvectors:

For λ_1

$$\ker\begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix} = \ker\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \Rightarrow \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

For λ_2

$$\ker\begin{bmatrix} 1 & -2 \\ 1 & -2 \end{bmatrix} = \ker\begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \Rightarrow \vec{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Now we can solve the initial value problem:

$$\vec{x}(0) = \begin{bmatrix} r(0) \\ w(0) \end{bmatrix} = \begin{bmatrix} 300 \\ 200 \end{bmatrix} = c_1\vec{v}_1 + c_2\vec{v}_2 = c_1\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$c_1 + 2c_2 = 300$, $c_1 + c_2 = 200$ has the solution $c_1 = c_2 = 100$.

$$\vec{x}(t) = \begin{bmatrix} r(t) \\ w(t) \end{bmatrix} = c_1\lambda_1^t\vec{v}_1 + c_2\lambda_2^t\vec{v}_2 = 100\,3^t\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 100\,4^t\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

i.e.

$$r(t) = 100\,(3^t + 2\,4^t)$$
$$w(t) = 100\,(3^t + 4^t)$$

(b) In the long run, what will be the proportion of rabbits to wolves?

The larger eigenvalue determines the long-term behavior:

$$\lim_{t\to\infty}\frac{r(t)}{w(t)} = \lim_{t\to\infty}\frac{3^t+2\,4^t}{3^t+4^t} = \lim_{t\to\infty}\frac{2\,4^t}{4^t} = 2$$

There will be twice as many rabbits as wolves.

80.5

(6) A rabbit population $r(t)$ and wolf population $w(t)$ are now modelled by the differential equations

$$\frac{dr}{dt} = 4r - 2w$$

$$\frac{dw}{dt} = r + w$$

The initial populations (as in the previous problem) are $r(0) = 300$ and $w(0) = 200$.

(a) Find formulas for $r(t)$ and $w(t)$

$$\vec{x}(t) = \begin{bmatrix} r(t) \\ w(t) \end{bmatrix} \qquad \frac{d\vec{x}}{dt} = A\vec{x}(t)$$

$$A = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}$$

The solution will be given by $\vec{x}(t) = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2$. We proceed directly to finding eigenvalues and eigenvectors:

$$\det(A - \lambda I) = \det\begin{bmatrix} 4-\lambda & -2 \\ 1 & 1-\lambda \end{bmatrix} = -(4-\lambda)(1-\lambda) + 2$$

$$= \lambda^2 - 5\lambda + 6 = 0 \Rightarrow \lambda = \tfrac{1}{2}\left(5 \pm \sqrt{25 - 4\cdot 6}\right) = \tfrac{1}{2}(5 \pm 1)$$

$\Rightarrow \lambda_i = 2, 3$ are the eigenvalues

Corresponding eigenvectors:

$\lambda_1 = 2$:

$$\ker\begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix} = \ker\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \Rightarrow \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\lambda_2 = 3$:

$$\ker\begin{bmatrix} 1 & -2 \\ 1 & -2 \end{bmatrix} = \ker\begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \Rightarrow \vec{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

We can now solve the initial value problem:

$$\vec{X}(0) = \begin{bmatrix} r(0) \\ w(0) \end{bmatrix} = \begin{bmatrix} 300 \\ 200 \end{bmatrix} = c_1 \vec{v}_1 + c_2 \vec{v}_2 = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

(Just as in the preceding problem). So $c_1 = c_2 = 100$.
The solution is

$$\vec{x}(t) = c_1 e^{\lambda t} \vec{v}_1 + c_2 e^{\lambda t} \vec{v}_2 = 100 e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 100 e^{3t} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

or

$$r(t) = 100 \left(e^{2t} + 2 e^{3t}\right)$$
$$w(t) = 100 \left(e^{2t} + e^{3t}\right)$$

(b) In the long run, what is the proportion of rabbits to wolves?
Again, the larger eigenvalue dominates as $t \to \infty$:

$$\lim_{t\to\infty} \frac{r(t)}{w(t)} = \lim_{t\to\infty} \frac{e^{2t} + 2e^{3t}}{e^{2t} + e^{3t}} = \lim_{t\to\infty} \frac{2e^{3t}}{e^{3t}} = \underline{2}.$$

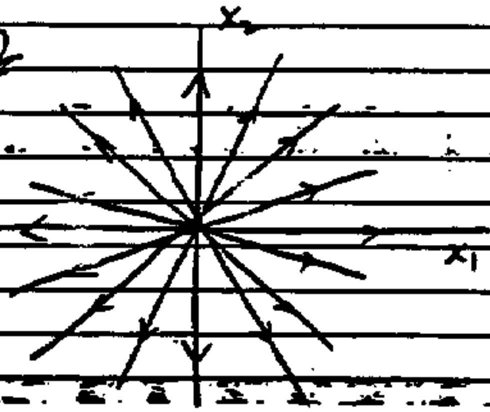
(7) Consider the system

$$\frac{dx_1}{dt} = a x_1 - b x_2$$
$$\frac{dx_2}{dt} = b x_1 + a x_2$$

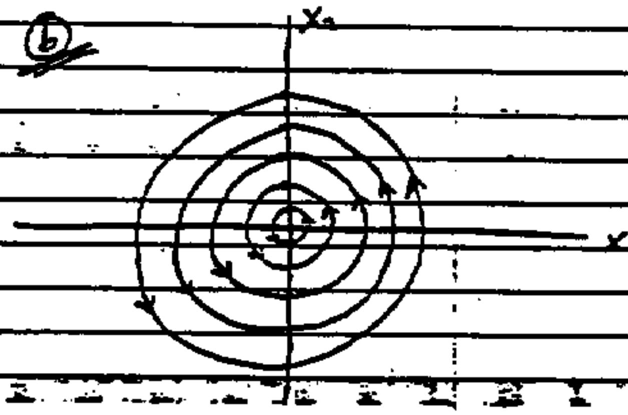
where a, b are constants.

(a) Sketch the phase portrait in the case $a=1, b=0$.
(b) " " " " " " " $a=0, b=1$

(a)

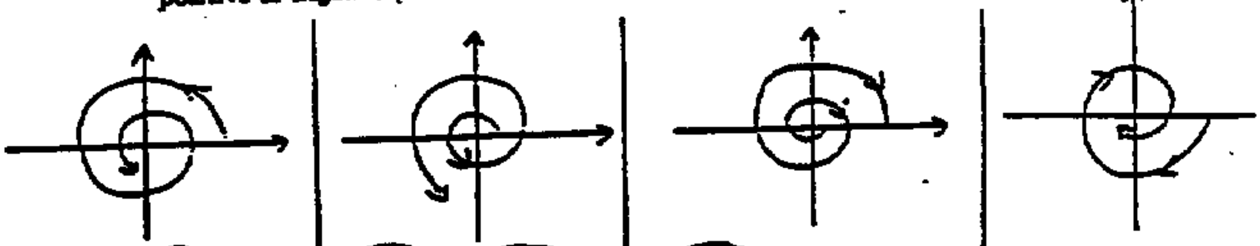


(b)



(7) continued

(c) For each of the four phase portraits below, indicate whether the constants a and b are positive or negative (circle the correct answers).



$a>0$	$b>0$	$a>0$	$b>0$	$a>0$	$b>0$	$a>0$	$b>0$
$a<0$	$b<0$	$a<0$	$b<0$	$a<0$	$b<0$	$a<0$	$b<0$

(8) Consider a herd of antelope in Serengeti Park. The model for instantaneous rate of change of the herd population is

- birth rate is 10% per year
- death rate ("natural causes") is 5% per year
- rate of predation proportional to square root of population — with proportionality constant equal to 1.5

(a) Set up a differential equation for the population $p(t)$:

$$\frac{dp}{dt} = \underbrace{+0.10\,p(t)}_{\text{birth}} - \underbrace{0.05\,p(t)}_{\text{death (n.c.)}} - \underbrace{1.5\,(p(t))^{\frac{1}{2}}}_{\text{death (predation)}}$$

so:

$$\frac{dp}{dt} = \frac{1}{20}p - \frac{3}{2}p^{\frac{1}{2}}$$

(b) Find the positive equilibrium solution for this differential equation and determine its stability:

$$\frac{dp}{dt} = \frac{1}{20}p - \frac{3}{2}p^{\frac{1}{2}} = 0 \Rightarrow p = 0 \text{ or } p_0 = (30)^2 = 900$$

80.8

The positive equilibrium solution is $p_0 = 900$.

Stability analysis: $\frac{dp}{dt} = f(p) \quad f(p_0) = 0$

$$\Rightarrow \frac{dp}{dt} = \underbrace{f(p_0)}_{0} + \frac{df}{dp}(p_0)\cdot(p-p_0) + \dots = \frac{df}{dp}(p_0)\cdot(p-p_0)$$

near $p = p_0$. If $\frac{df}{dp}(p_0) > 0$ the equilibrium point is unstable, if $\frac{df}{dp}(p_0) < 0$ it is stable.

In our case

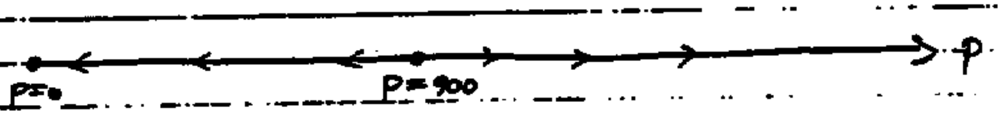
$$\frac{df}{dp}(p_0) = \frac{1}{20} - \frac{3}{2}\frac{1}{2}p_0^{-\frac{1}{2}} = \frac{1}{20} - \frac{3}{4\cdot 30} = \frac{1}{40} > 0$$

So the equilibrium population at $p_0 = 900$ is unstable.

(c) Describe the long-term behavior of the population and explain how this behavior depends on the initial population:

If $p < p_0 = 900 \quad \frac{dp}{dt} < 0$ and $\lim_{t\to\infty} p(t) = 0$

If $p > p_0 = 900 \quad \frac{dp}{dt} > 0$ and $\lim_{t\to\infty} p(t) = \infty$



(9) In the recent past, the real income of an average American family has doubled about every 30 years. However, this rate of growth has slowed, and at the current rate it would take 200 years for the real family income to double. At this current rate, by how many percent does the income grow in a generation (30 years)?

Let r be the rate at which the income x grows each year: in a discrete model:

$$x(t+1) = x(t) + r\,x(t)$$

c. $X(t) = (1+r)^t X(0)$

From the information given, $(1+r)^{200} = 2$.
Since r is so small, to very good approximation

$$(1+r)^{200} = 1+200r+\cdots = 2 \implies r \cong \frac{1}{200}$$

After 30 years, $X(t) \cong \left(1+\frac{1}{200}\right)^{30} X(0) \cong X(0) + \frac{30}{200}X(0) + \cdots$. So the income has grown by a factor of approximately $\frac{3}{20}$, or 15%

(10) Let V be the vector space of quadratic forms in two variables, i.e. functions of the form $f(x_1,x_2) = ax_1^2 + bx_1x_2 + cx_2^2$.
Consider the linear transformation $T: V \to V$

$$T(f) = \frac{\partial f}{\partial x_1}x_2 - \frac{\partial f}{\partial x_2}x_1$$

(a) Find the matrix of T with respect to the basis $\{x_1^2, x_1x_2, x_2^2\}$:

Look at the action of T on the basis:

$$T(x_1^2) = \frac{\partial x_1^2}{\partial x_1}x_2 - \frac{\partial x_1^2}{\partial x_2}x_1 = 2x_1x_2 - 0 = 2x_1x_2$$

$$T(x_1x_2) = \frac{\partial(x_1x_2)}{\partial x_1}x_2 - \frac{\partial(x_1x_2)}{\partial x_2}x_1 = x_2^2 - x_1^2 = -x_1^2 + x_2^2$$

$$T(x_2^2) = \frac{\partial x_2^2}{\partial x_1}x_2 - \frac{\partial x_2^2}{\partial x_2}x_1 = 0 - 2x_2x_1 = -2x_1x_2$$

This gives the columns of the matrix A of T with respect to this basis:

$$A = \begin{bmatrix} 0 & -1 & 0 \\ 2 & 0 & -2 \\ 0 & 1 & 0 \end{bmatrix}$$

10(b) Find bases for the kernel and range of T:

Proceed by row-reducing A:

$$\begin{bmatrix} 0 & -1 & 0 \\ 2 & 0 & -2 \\ 0 & 1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

most general element of ker A: $\begin{bmatrix} t \\ 0 \\ t \end{bmatrix}$ basis: $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

This represents the quadratic form:

$x_1^2 + x_2^2$ basis for ker T

The basis for the range of A is given by the columns of A corresponding to the leading variables, i.e.

$$\left\{ \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

These represent the quadratic forms

$\{2x_1x_2, -x_1^2 + x_2^2\}$ basis for range (T)

(11) appears on the next page

20

- The temperature in the $x_1 - x_2$ plane is given by $T(x_1, x_2) = 30 - 3x_1^2 - 3x_2^2 - 2x_1x_2$ (in degrees Celsius). Some heat loving bugs crawl always in the direction in which the temperature increases most rapidly, i.e., in the direction of the gradient of T. Which of the sketches below best represents possible trajectories of these bugs? Justify your choice!

(Circle)

justification: $\frac{d\vec{x}}{dt} = \vec{\nabla} T$

i.e.,

$\frac{dx_1}{dt} = \frac{\partial T}{\partial x_1} = -6x_1 - 2x_2$

$\frac{dx_2}{dt} = \frac{\partial T}{\partial x_2} = -2x_1 - 6x_2$

so $\frac{d\vec{x}}{dt} = A\vec{x}$

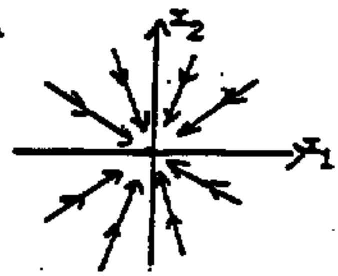
$A = \begin{bmatrix} -6 & -2 \\ -2 & -6 \end{bmatrix}$

eigenvalues are $-4, -8$

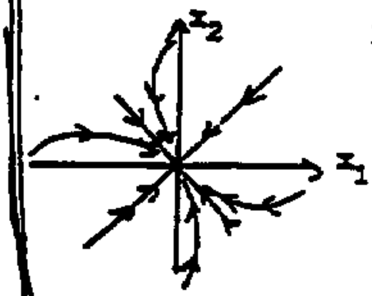
$\Rightarrow$ stable node.

To distinguish possibility G from B, pick a point on either axis and see where it goes. This rules out G, the other stable node.

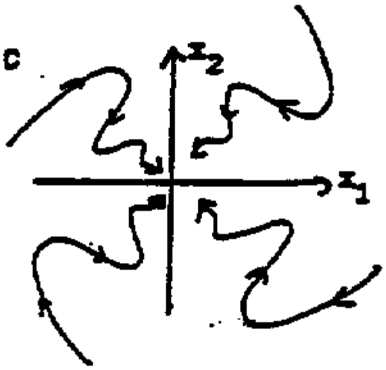
A



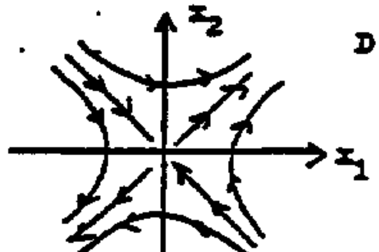
B



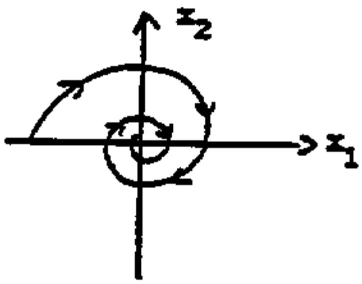
C



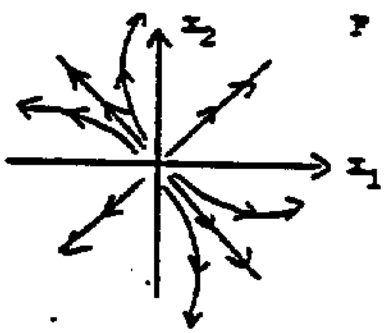
D



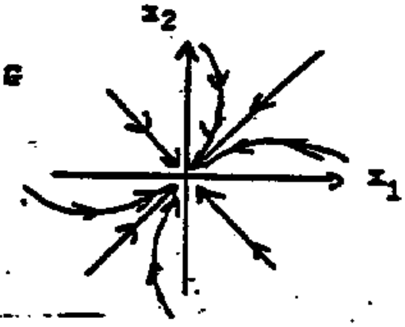
E



F



G



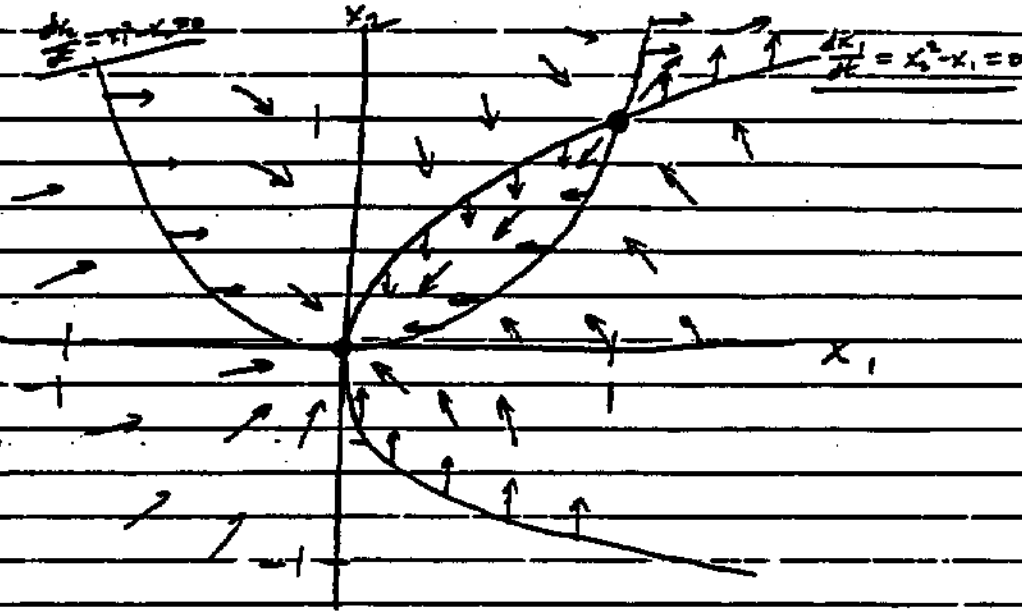
80.12

(12) Consider the system

$$\frac{dx_1}{dt} = x_2^2 - x_1$$

$$\frac{dx_2}{dt} = x_1^2 - x_2$$

(a) Perform a qualitative phase plane analysis, i.e. find null-clines, equilibrium points, general directions of vector field.



(b) List equilibrium points and determine their stability

The Jacobian is $J(x_1, x_2) = \begin{bmatrix} -1 & 2x_2 \\ 2x_1 & -1 \end{bmatrix}$

<u>At (0,0)</u>: $J(0,0) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$: <u>stable node</u>

(This is a case of degenerate eigenvalues – all vectors near (0,0) are eigenvectors of $J(0,0)$.)

<u>At (1,1)</u>: $J(1,1) = \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix}$ <u>saddle point</u>

So, <u>unstable</u>. [Eigenvalues are $1, -3$; eigenvectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ resp.]

(13) Consider the vector space V consisting of all real 2×2 matrices for which the vector $\begin{bmatrix}1\\2\end{bmatrix}$ is an eigenvector. Find a basis for V and determine its dimension:

If $\begin{bmatrix}1\\2\end{bmatrix}$ is an eigenvector of $A=\begin{bmatrix}a&b\\c&d\end{bmatrix}$, then

$$\begin{bmatrix}a&b\\c&d\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix}=\lambda\begin{bmatrix}1\\2\end{bmatrix}\Rightarrow \begin{array}{l}a+2b=\lambda\\c+2d=2\lambda\end{array}$$

so

$$A=\begin{bmatrix}\lambda-2b & b\\ 2\lambda-2d & d\end{bmatrix}$$

is the most general possibility. A possible basis for V is therefore (e.g. setting $b=1, d=0, \lambda=0$, and so on):

$$\left\{\begin{bmatrix}-2&1\\0&0\end{bmatrix},\begin{bmatrix}0&0\\-2&1\end{bmatrix},\begin{bmatrix}1&0\\2&0\end{bmatrix}\right\}$$

V is therefore 3-dimensional.

(14) Consider the van der Pol equation

$$\frac{d^2x}{dt^2}=a(1-x^2)\frac{dx}{dt}-x$$

where $a>0$ is a constant.

(a) Transform this second-order differential equation into a system of two first-order differential equations:

$$\begin{cases}\dfrac{dx}{dt}=v & \text{(basically defines } v\text{)}\\[2ex] \dfrac{dv}{dt}=-x+a(1-x^2)v\end{cases}$$

15(b) Linearize this system at the equilibrium point.

Drop the x^2 term; more formally, the Jacobian is

$$J(x,v) = \begin{bmatrix} 0 & 1 \\ -1-2axv & a(1-x^2) \end{bmatrix}$$

$$J(0,0) = \begin{bmatrix} 0 & 1 \\ -1 & a \end{bmatrix}$$

so the linearized system is

$$\begin{cases} \frac{dx}{dt} = v \\ \frac{dv}{dt} = -x + av \end{cases}$$

(c) Determine the stability of (0,0) — Does the answer depend on the value of a?

$\det J(0,0) = 1 > 0$
$\operatorname{tr} J(0,0) = a > 0$

So (0,0) is unstable, for all values of a

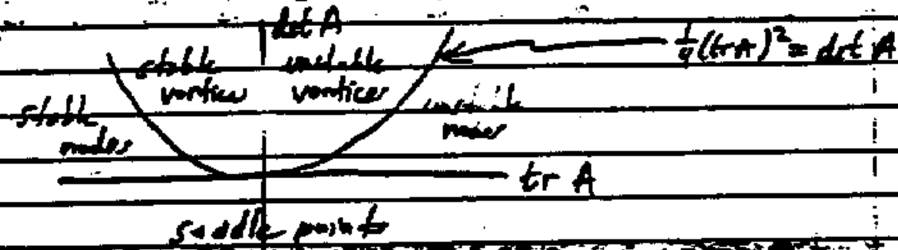
(d) Whether (0,0) is an unstable node or an unstable vortex does depend on a:

$\frac{1}{4}(\operatorname{tr} J(0,0))^2 = \frac{1}{4}a^2 \qquad \det J(0,0) = 1$

If $\frac{1}{4}a^2 > 1$, i.e. $a > 2$: unstable node

If $\frac{1}{4}a^2 < 1$, i.e. $a < 2$: unstable vortex

Remember:



80.15