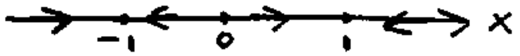
(VII) : SOLUTIONS

TRUE [✓ FALSE] $x = 0$ is a stable equilibrium solution of the differential equation $\frac{dx}{dt} = x - x^3$.



TRUE [✓ FALSE] If A is an 8×5 matrix, then the kernel of A is at least three-dimensional.

Counter example: If $A = \begin{bmatrix} I_5 \\ \hline \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \end{bmatrix}$, then kernel(A) = 0

[✓ TRUE] FALSE All trajectories of the dynamical system $\vec{x}(t+1) = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \vec{x}(t)$ approach $\vec{0}$ as t goes to infinity.

The matrix $\begin{bmatrix} p & -q \\ q & p \end{bmatrix}$ represents a rotation-dilation, with dilation factor $\sqrt{p^2+q^2}$. In our case, the dilation factor is less than 1 : The transformation $\vec{x}(t+1) = A\vec{x}(t)$ represents a rotation-contraction. The trajectories spiral i

TRUE [✓ FALSE] There exist matrices A in $\mathbb{R}^{5\times4}$ and B in $\mathbb{R}^{4\times5}$ such that AB is invertible.

The kernel of B is non-zero, since the system $B\vec{x} = \vec{0}$ has less equations than variables. The kernel of AB is non-zero as well (pick a non-zero $\vec{x}$ such that $B\vec{x} = \vec{0}$; then, $AB\vec{x} = \vec{0}$). Therefore, AB is non-invertible.

[✓ TRUE] FALSE Consider the vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ in $\mathbb{R}^m$. Let A be a $p \times m$ matrix. If the vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ are linearly dependent, then so are the vectors $A\vec{v}_1, A\vec{v}_2, \ldots, A\vec{v}_n$.

One of the $\vec{v}_i$'s will be a linear combination of the others, for example, $\vec{v}_n = c_1\vec{v}_1 + \ldots + c_{n-1}\vec{v}_{n-1}$. Then, $A\vec{v}_n = A(c_1\vec{v}_1+\ldots+c_{n-1}\vec{v}_{n-1}) = c_1A\vec{v}_1 + \ldots + c_{n-1}A\vec{v}_{n-1}$. This shows that the vectors $A\vec{v}_1, \ldots , A\vec{v}_n$ are linearly dependent.

[✓ TRUE] FALSE If S and A are orthogonal $n \times n$ matrices, then the matrix $S^{-1}AS$ is orthogonal as well.

Products and inverses of orthogonal matrices are orthogonal.

TRUE [✓ FALSE] Orthogonal matrices are diagonalizable (over the reals).

Consider a rotation matrix as a counterexample.

TRUE [✓ FALSE] The differential equation $\frac{d^2x}{dt^2} + 7\frac{dx}{dt} + 13x = 0$ has no real solutions $x(t)$.

x(t) = 0 is a real solution (there are many more. In fact, the real solutions form a 2-dimensional linear space)

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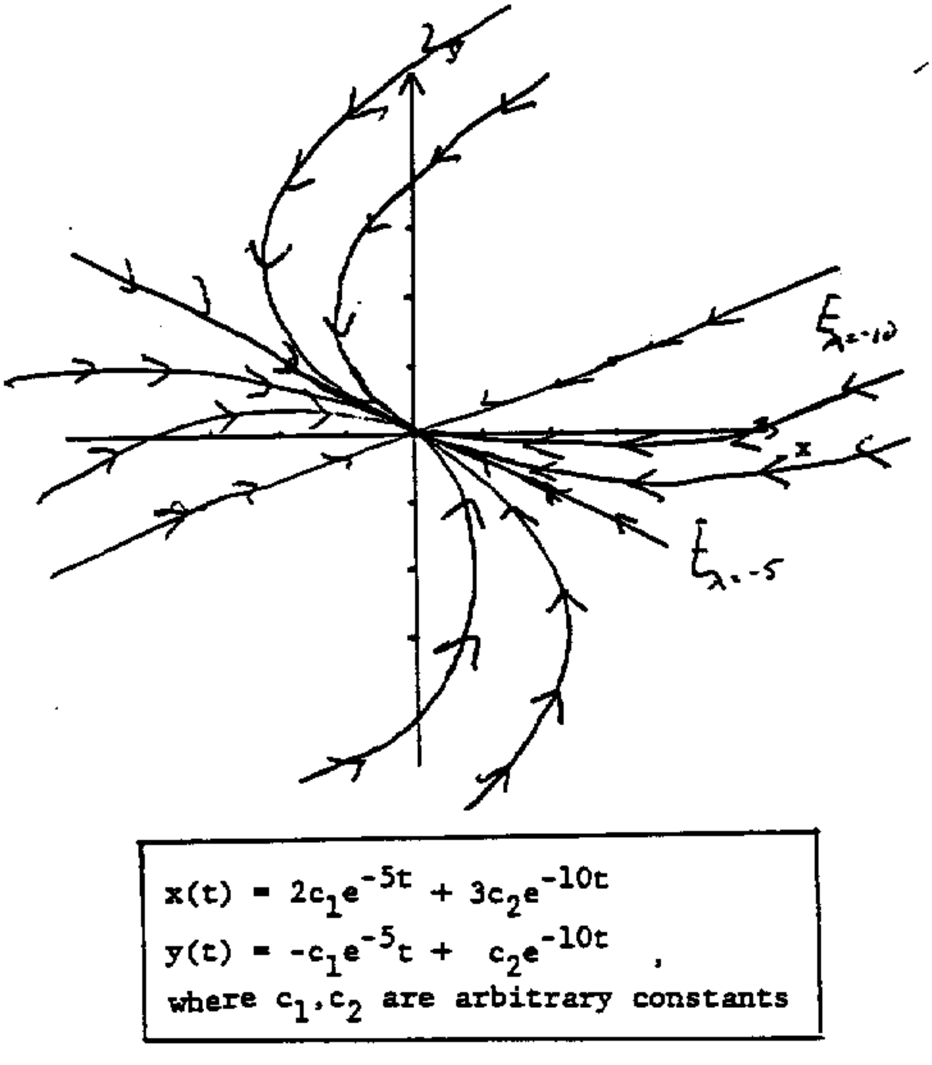
2. Find all solutions of the system $\left| \begin{array}{l} \frac{dx}{dt} = -8x - 6y \\ \frac{dy}{dt} = -\ x - 7y \end{array} \right|$ and sketch a phase portrait for this system.

$\lambda^2 + 15\lambda + 50 = 0 = (\lambda+5)(\lambda+10)$ eigenvalues : $-5, -10$

$E_{-5} =$ kernel $\begin{bmatrix} -3 & -6 \\ -1 & -2 \end{bmatrix} =$ span $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$E_{-10} =$ kernel $\begin{bmatrix} 2 & -6 \\ -1 & 3 \end{bmatrix} =$ span $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$

general solution $\vec{x}(t) = c_1 e^{-5t} \begin{bmatrix} 2 \\ -1 \end{bmatrix} + c_2 e^{-10t} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$



$x(t) = 2c_1e^{-5t} + 3c_2e^{-10t}$

$y(t) = -c_1e^{-5}t + c_2e^{-10t}$,

where c_1, c_2 are arbitrary constants

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3. **Find the solution of the initial value problem**

$$\frac{d^2x}{dt^2} - 6\frac{dx}{dt} + 25x = 0\,, \quad x(0) = 0\,, \quad x\left(\frac{\pi}{8}\right) = 1$$

Draw a rough sketch of your solution.

Trial solution $x(t) = e^{\lambda t}$

$\lambda^2 - 6\lambda + 25 = 0 \qquad \lambda_{1,2} = \frac{6 \pm \sqrt{36-100}}{2} = 3 \pm 4i$

Find real and imaginary parts of the complex solution $z(t) = e^{(3+4i)t}$

$z(t) = e^{(3+4i)t} = e^{3t+4it} = e^{3t}\cos 4t + i\, e^{3t}\sin 4t$

General real solution: $\underline{x(t) = c_1 e^{3t}\cos 4t + c_2 e^{3t}\sin 4t}$

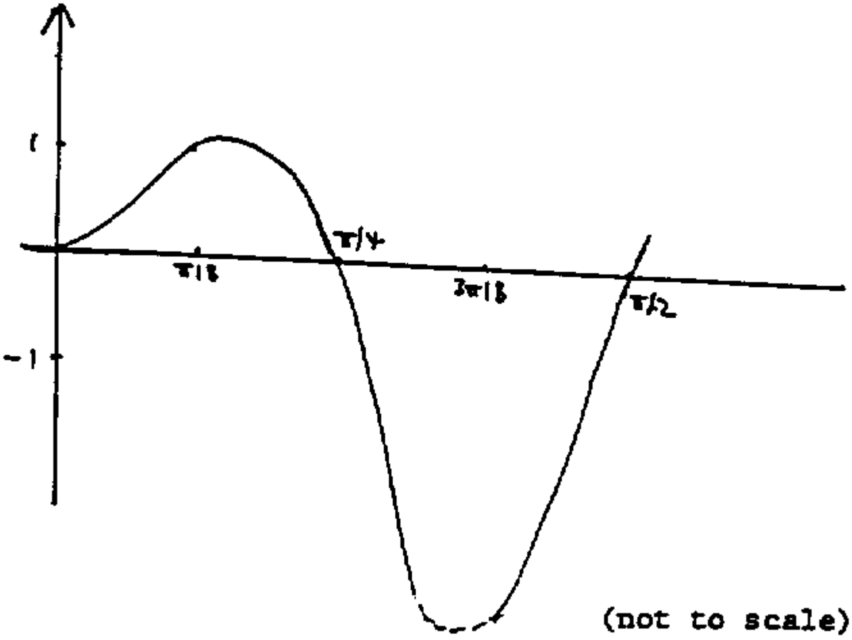
Find c_1 and c_2 such that initial conditions are satisfied.

$x(0) = 0$ means that $c_1 = 0$:

$x(t) = c_2 e^{3t}\sin 4t$

$x\left(\frac{\pi}{8}\right) = c_2 e^{3\pi/8}\sin\left(\frac{\pi}{2}\right) = c_2 e^{3\pi/8} = 1 \quad \Rightarrow \quad c_2 = e^{-3\pi/8}$

$$\boxed{x(t) = e^{-\frac{3\pi}{8}} \cdot e^{3t} \cdot \sin 4t = e^{3(t-\frac{\pi}{8})}\sin 4t}$$



(not to scale)

2B

4. "A certain person buys sheep, goats, and hogs, to the number of 100, for 100 crowns; the sheep cost him $\frac{1}{2}$ a crown a-piece; the goats $1\frac{1}{3}$ crown; and the hogs, $3\frac{1}{2}$ crowns. How many had he of each?" (From Leonhard Euler: *Elements of Algebra*, St. Petersburg, 1770. Translated by Rev. John Hewlett.)
Find *all* solutions to this problem.

x = # sheep, y = # goats, z = # hogs.

$$\left|\begin{matrix} x+y+z=100 \\ \frac{x}{2}+\frac{4}{3}y+\frac{7}{2}z=100 \end{matrix}\right| \quad \left[\begin{matrix} 1 & 1 & 1 \\ \frac{1}{2} & \frac{4}{3} & \frac{7}{2} \end{matrix}\middle|\begin{matrix} 100 \\ 100 \end{matrix}\right] \longrightarrow \left[\begin{matrix} 1 & 0 & -\frac{13}{5} \\ 0 & 1 & \frac{13}{5} \end{matrix}\middle|\begin{matrix} 40 \\ 60 \end{matrix}\right]$$

$$\begin{matrix} x & & -\frac{13}{5}z = 40 \\ & y & +\frac{18}{5}z = 60 \end{matrix} \qquad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 40+\frac{13}{5}t \\ 60-\frac{18}{5}t \\ t \end{bmatrix}$$

The three quantities x,y,z must be non-negative integers. The formulas $x = 40 + \frac{13}{5}t$ and $y = 60-\frac{18}{5}t$ imply that t must be divisible by 5 , so t = 0, 5, 10, 15, 20 , ... Also, y must be non-negative, so t = 15 (note that y = 60-54 = 6 for t=15 , but y = 60-72 = -12 for t=20).

Number of sheep	Number of goats	Number of hogs
40	60	0
53	42	5
66	24	10
79	6	15

5. Find a matrix A such that kernel(A) = span $\left(\underset{\vec{v}}{\begin{bmatrix}1\\1\\1\\1\end{bmatrix}}, \underset{\vec{w}}{\begin{bmatrix}1\\2\\3\\4\end{bmatrix}}\right)$.

The matrix A must have 4 columns so that the products $A\vec{v}$ and $A\vec{w}$ are defined. The <u>rows</u> of A must be of the form $[a\ b\ c\ d]$, with $[a\ b\ c\ d]\vec{v} = 0$ and $[a\ b\ c\ d]\vec{w} = 0$, i.e.,

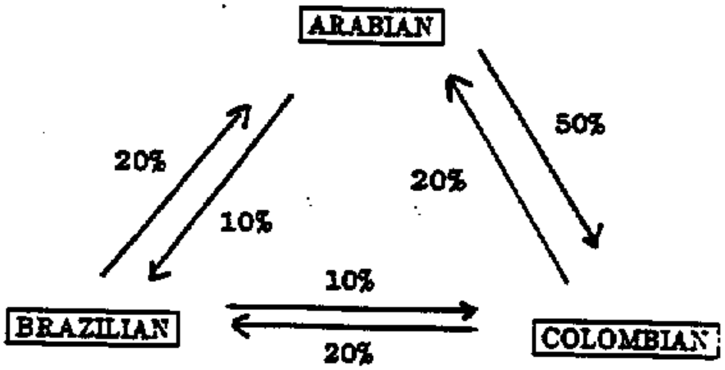
$$\left|\begin{matrix} a+b+c+d=0 \\ a+2b+3c+4d=0 \end{matrix}\right| \text{, or } \left|\begin{matrix} a & & -c-2d=0 \\ & b & +2c+3d=0 \end{matrix}\right|$$

$$\begin{aligned} [a\ b\ c\ d] &= [s+2t \quad -2s-3t \quad s \quad t], \text{ where } s,t \text{ are arbitrary} \\ &= s[1\ \ -2\ \ 1\ \ 0] + t[2\ \ -3\ \ 0\ \ 1] \end{aligned}$$

Set $A = \begin{bmatrix} 1 & -2 & 1 & 0 \\ 2 & -3 & 0 & 1 \end{bmatrix}$. <u>There are many other solutions</u>.

$$\boxed{A = \begin{bmatrix} 1 & -2 & 1 & 0 \\ 2 & -3 & 0 & 1 \end{bmatrix}}$$

6. A market research organization is studying a large group of coffee lovers who buy a bag of coffee each week. The choices are Arabian, Brazilian, and Colombian beans. Let $a(t)$, $b(t)$, and $c(t)$ be the fractions of the people in the sample who prefer the Arabian, the Brazilian, and the Colombian beans, t weeks after the beginning of the study. From one week to the next, people change their choice as indicated in the diagram below.



For example, 10% of the people who bought the Arabian beans a week ago will buy the Brazilian beans today, and 50% will switch to the Colombian beans, while the remaining 40% will again buy the Arabian beans.

(a) We introduce the state vector $\vec{x}(t) = \begin{bmatrix} a(t) \\ b(t) \\ c(t) \end{bmatrix}$. Find the matrix A such that $\vec{x}(t+1) = A\vec{x}(t)$. Time t is measured in weeks.

$$a(t+1) = 0.4a(t) + 0.2b(t) + 0.2c(t)$$

only 40% of the people who bought Arabian coffee will again make the same choice, since 10% switch to Brazilian and 50% to Columbian

20% of the people who bought Brazilian or Columbian coffee a week ago swith to Arabian

Likewise:

$$b(t+1) = 0.1a(t) + 0.7b(t) + 0.2c(t)$$
$$c(t+1) = 0.5a(t) + 0.1b(t) + 0.6c(t)$$

$$A = \begin{bmatrix} .4 & .2 & .2 \\ .1 & .7 & .2 \\ .5 & .1 & .6 \end{bmatrix}$$

continued

6, continued

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(b) We are told that $\begin{bmatrix} 3 \\ 1 \\ -4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 5 \\ 7 \\ 8 \end{bmatrix}$ is an eigenbasis for A. In the long run, which fraction of the people in the sample will buy each of the types of beans?

To find the eigenvalues, compute $A\vec{v}$ for the three eigenvectors above.

$$A\begin{bmatrix} 3 \\ 1 \\ -4 \end{bmatrix} = \begin{bmatrix} 0.4 & 0.2 & 0.2 \\ 0.1 & 0.7 & 0.2 \\ 0.5 & 0.1 & 0.6 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \\ -4 \end{bmatrix} = \begin{bmatrix} 0.6 \\ 0.2 \\ -0.8 \end{bmatrix} = 0.2\begin{pmatrix} 3 \\ 1 \\ -4 \end{pmatrix}$$

The eigenvalue is 0.2

In the same way, find the eigenvalues for the two other eigenvectors.

$$\begin{bmatrix} 3 \\ 1 \\ -4 \end{bmatrix} \leftrightarrow 0.2 \qquad \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \leftrightarrow 0.5 \qquad \begin{bmatrix} 5 \\ 7 \\ 8 \end{bmatrix} \leftrightarrow 1$$

If $\vec{x}(0) = c_1\begin{bmatrix} 3 \\ 1 \\ -4 \end{bmatrix} + c_2\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + c_3\begin{bmatrix} 5 \\ 7 \\ 8 \end{bmatrix}$, then

$$\vec{x}(t) = c_1 0.2^t\begin{bmatrix} 3 \\ 1 \\ -4 \end{bmatrix} + c_2\ 0.5^t\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + c_3\begin{bmatrix} 5 \\ 7 \\ 8 \end{bmatrix}$$

The first two summands approach $\vec{0}$ as t goes to infinity, so $\vec{x}(t) \approx c_3\begin{bmatrix} 5 \\ 7 \\ 8 \end{bmatrix}$

The beans will be bought in the proportion 5 : 7 : 8 , or 25% : 35% : 40% .

(c) Suppose that at the beginning of the study (at $t = 0$) everybody chooses the Arabian beans. Which fraction of the people in the study will choose the Arabian beans after t weeks?

We are told that $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 0.25\begin{bmatrix} 3 \\ 1 \\ -4 \end{bmatrix} - 0.6\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + 0.05\begin{bmatrix} 5 \\ 7 \\ 8 \end{bmatrix}$.

$$\vec{x}(t) = 0.25 \cdot 0.2^t\begin{bmatrix} 3 \\ 1 \\ -4 \end{bmatrix} - 0.6 \cdot 0.5^t\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + 0.05\begin{bmatrix} 5 \\ 7 \\ 8 \end{bmatrix}$$

$a(t)$ is the first component:

$$\boxed{a(t) = 0.75 \cdot 0.2^t + 0.25}$$

7. Find a 2×2 matrix A with all of the following properties:
   - (a) $\det(A) = 2$
   - (b) $\text{trace}(A) = 3$
   - (c) The vectors $\begin{bmatrix}2\\5\end{bmatrix}$ and $\begin{bmatrix}1\\3\end{bmatrix}$ are eigenvectors of A.

We can find the eigenvalues of A:

$$\lambda^2 - 3\lambda + 2 = (\lambda-1)(\lambda-2) = 0 \qquad \lambda = 1, 2$$

(Recall that the characteristic polynomial is $\lambda^2 - (\mathrm{Tr}A)\lambda + \det A = 0$)

One possibility is to construct A so that $\begin{bmatrix}2\\5\end{bmatrix}$ is an eigenvector w. eigenvalue 1, and $\begin{bmatrix}1\\3\end{bmatrix}$ with eigenvalue 2.

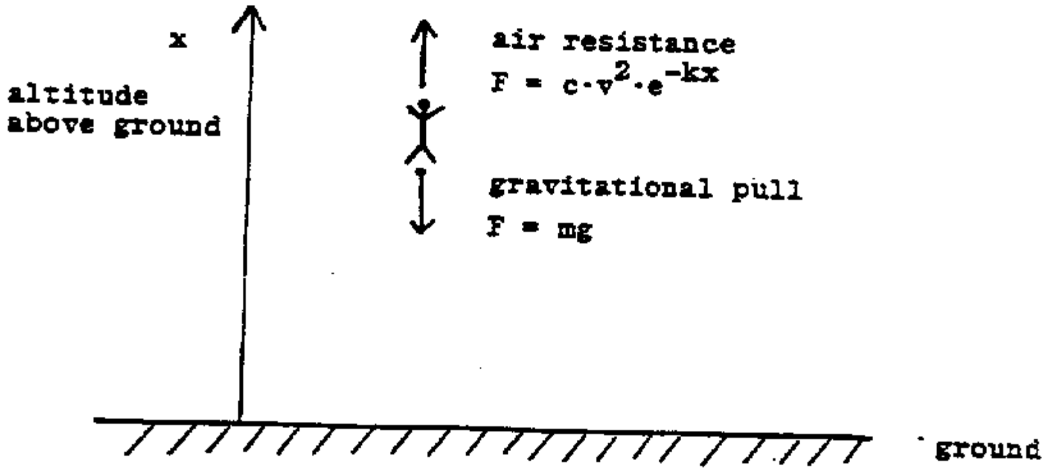
$S^{-1}AS = D$, or $A = SDS^{-1}$, with $S = \begin{bmatrix}2 & 1\\5 & 3\end{bmatrix}$, $D = \begin{bmatrix}1 & 0\\0 & 2\end{bmatrix}$

$$A = \begin{bmatrix}2 & 1\\5 & 3\end{bmatrix}\begin{bmatrix}1 & 0\\0 & 2\end{bmatrix}\begin{bmatrix}3 & -1\\-5 & 2\end{bmatrix} = \begin{bmatrix}-4 & 2\\-15 & 7\end{bmatrix}$$

$A = \begin{bmatrix}-4 & 2\\-15 & 7\end{bmatrix}$ Another correct answer: $\begin{bmatrix}7 & -2\\15 & -4\end{bmatrix}$

8. Consider a skydiver falling vertically from a high altitude. Two forces are acting on her body:

- The force of gravity, mg, where m is her mass.
- The force of air resistance, which is assumed to be proportional to the square of her speed v, and also to the density of the air, which decreases exponentially with altitude.



Note that the coordinates are chosen in such a way that the resistance force is positive, and the gravitational force is negative.

8, continued

(a) Set up a second order differential equation describing the motion of the sky diver. (Your formula will contain the constants c, k, m, and g.)

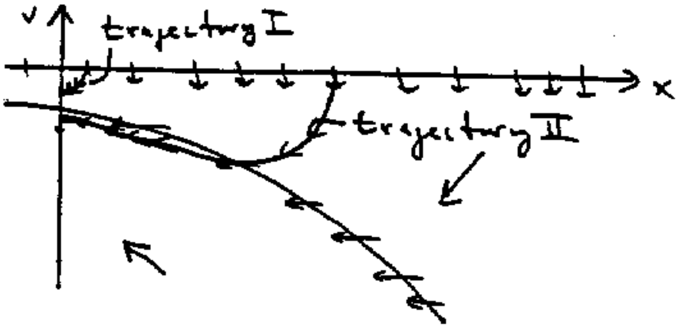
$$F = ma = m\frac{d^2x}{dt^2} = \underbrace{cv^2e^{-kx}}_{\text{resistance}} - \underbrace{mg}_{\text{weight}} = c\left(\frac{dx}{dt}\right)^2 e^{-kx} - mg$$

$$\boxed{\frac{d^2x}{dt^2} = \frac{c}{m}\left(\frac{dx}{dt}\right)^2 e^{-kx} - g}$$

(b) Convert your solution, in (a), into a system of two first order differential equations.

$$\boxed{\begin{aligned} \frac{dx}{dt} &= v \\ \frac{dv}{dt} &= \frac{c}{m}v^2e^{-kx} - g \end{aligned}}$$

(c) Perform the qualitative phase plane analysis on the system you found in (b). We are only interested in the case when x is positive and v is negative (the sky diver is falling!). Therefore, you need to consider the fourth quadrant only.



$\frac{dx}{dt} = 0$ for $v = 0$ (x-axis)

$\frac{dv}{dt} = \frac{c}{m}v^2e^{-kx} - g = 0$ for

$v^2 = \frac{gm}{c}e^{kx}$

$v = -\sqrt{\frac{gm}{c}}\,e^{\frac{kx}{2}}$

(exponential Function)

(d) Consider a trajectory starting at a point with $v = 0$. Relate the features of this trajectory to the fall of the skydiver.

Initially, the sky diver picks up speed quickly. As the speed increases and the body falls into denser layers of the atmosphere, the magnitude of the resistance increases, and the force of the resistance may eventually become stronger than the force of gravity (the weight), in which case the sky diver will slow down. This happens only if the sky diver is dropped from a large height (trajectory II in c), but not trajectory I).

9. Consider the matrix $A = \begin{bmatrix} k & 0 & 0 & 0 & 1 \\ 0 & k & 0 & 0 & 1 \\ 0 & 0 & k & 0 & 1 \\ 0 & 0 & 0 & k & 1 \\ 1 & 1 & 1 & 1 & k \end{bmatrix}$, where k is a constant.

(a) For which choice(s) of the constant k is the last column of A a linear combination of the first four columns?

Want $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ k \end{bmatrix} = c_1 \begin{bmatrix} k \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ k \\ 0 \\ 0 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 0 \\ k \\ 0 \\ 1 \end{bmatrix} + c_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ k \\ 1 \end{bmatrix}$

Looking at the first four components we realize that we must have $c_1 = c_2 = c_3 = c_4 = \frac{1}{k}$ (in particular, $k \neq 0$)

For the last component:

$$k = 4 \cdot \frac{1}{k} \Rightarrow k^2 = 4 \Rightarrow \boxed{k = \pm 2}$$

(b) For which choice(s) of the constant k is the matrix A invertible?

Either compute the determinant (it is $k^3(k^2-4)$) or use part (a) to find

$$\boxed{A \text{ is invertible if } k \neq 0, 2, -2}$$

(c) Consider the matrix $M = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}$. Find as many linearly independent eigenvectors for M as you can.

A is symmetric, and therefore diagonalizable: There are 5 linearly independent eigenvectors.

λ is an eigenvalue of M if $(M - \lambda I_n)$ is not invertible.

In (b) we have seen that this is the case for $\lambda = 0, 2, -2$.

$\boxed{\text{eigenvalues : } 0, 2, -2}$

$$E_0 = \text{kernel} \begin{bmatrix} 0&0&0&0&1 \\ 0&0&0&0&1 \\ 0&0&0&0&1 \\ 0&0&0&0&1 \\ 1&1&1&1&0 \end{bmatrix} = \text{kernel} \begin{bmatrix} 1&1&1&1&0 \\ 0&0&0&0&1 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \end{bmatrix} \quad \begin{matrix} x_1+x_2+x_3+x_4 = 0 \\ x_5 = 0 \end{matrix} \quad \begin{bmatrix} -r-s-t \\ r \\ s \\ t \\ 0 \end{bmatrix} = r \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t$$

basis of E_0

$$E_2 = \text{kernel} \begin{bmatrix} -2&0&0&0&1 \\ 0&-2&0&0&1 \\ 0&0&-2&0&1 \\ 0&0&0&-2&1 \\ 1&1&1&1&-1 \end{bmatrix} = \text{kernel} \begin{bmatrix} 1&0&0&0&-\frac{1}{2} \\ 0&1&0&0&-\frac{1}{2} \\ 0&0&1&0&-\frac{1}{2} \\ 0&0&0&1&-\frac{1}{2} \\ 0&0&0&0&0 \end{bmatrix} \quad \begin{matrix} x_1 - \frac{1}{2}x_5 = 0 \\ x_2 - \frac{1}{2}x_5 = 0 \\ x_3 - \frac{1}{2}x_5 = 0 \\ x_4 - \frac{1}{2}x_5 = 0 \end{matrix} \quad \begin{bmatrix} t/2 \\ t/2 \\ t/2 \\ t/2 \\ t \end{bmatrix} = t \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \\ 1 \end{bmatrix}$$

basis of E_2

$$E_{-2} = \text{kernel} \begin{bmatrix} 2&0&0&0&1 \\ 0&2&0&0&1 \\ 0&0&2&0&1 \\ 0&0&0&2&1 \\ 1&1&1&1&3 \end{bmatrix} = \text{kernel} \begin{bmatrix} 1&0&0&0&\frac{1}{2} \\ 0&1&0&0&\frac{1}{2} \\ 0&0&1&0&\frac{1}{2} \\ 0&0&0&1&\frac{1}{2} \\ 0&0&0&0&0 \end{bmatrix} \quad \begin{matrix} x_1 + \frac{x_5}{2} = 0 \\ x_2 + \frac{x_5}{2} = 0 \\ x_3 + \frac{x_5}{2} = 0 \\ x_4 + \frac{x_5}{2} = 0 \end{matrix} \quad \begin{bmatrix} -t/2 \\ -t/2 \\ -t/2 \\ -t/2 \\ t \end{bmatrix} = t \begin{bmatrix} -1/2 \\ -1/2 \\ -1/2 \\ -1/2 \\ 1 \end{bmatrix}$$

basis of E_{-2}

$$\boxed{\text{eigenbasis: } \begin{bmatrix}-1\\1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}-1\\0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}-1\\0\\0\\1\\0\end{bmatrix}, \begin{bmatrix}1/2\\1/2\\1/2\\1/2\\1\end{bmatrix}, \begin{bmatrix}-1/2\\-1/2\\-1/2\\-1/2\\1\end{bmatrix}}$$

10. The interaction of two species of animals is modelled by the differential equations

$$\left| \begin{aligned} \frac{1}{x}\frac{dx}{dt} &= 1 - x + by - b \\ \frac{1}{y}\frac{dy}{dt} &= 1 - y + bx - b \end{aligned} \right|$$

where b is a constant, different from 1 and -1.

(a) What does the sign of the constant b tell you about the way the two species interact?

Symbiosis for positive b , and competition for negative b .

(b) The system above has exactly one equilibrium point in the first quadrant, i.e., with $x > 0$ and $y > 0$. Find this equilibrium point.

$$\left| \begin{matrix} 1 - x + by - b = 0 \\ 1 - y + bx - b = 0 \end{matrix} \right| \quad \left| \begin{matrix} x - by = 1-b \\ -bx + y = 1-b \end{matrix} \right| + bI \quad \left. \begin{matrix} x - by = 1-b \\ (1-b^2)y = (1-b^2) \end{matrix} \right\} \Rightarrow \boxed{x = y = 1}$$

(c) Linearize the system at the equilibrium point you found in b) .

$$J(x,y) = \begin{bmatrix} 1-2x+by-b & bx \\ by & 1-2y+bx-b \end{bmatrix}$$

$$J(1,1) = \begin{bmatrix} -1 & b \\ b & -1 \end{bmatrix}$$

(d) Determine the nature of the equilibrium point (saddle, node, ...), and its stability. Your answer will depend on the constant b.

The matrix J(1,1) is symmetric and invertible, so the equilibrium point will be a node or a saddle. It's a saddle if $\det J = 1 - b^2 < ($ i.e, if $|b|>1$. Otherwise, it's a stable node (the trace is negativ

Alternatively, find the eigenvalues: $\lambda^2+2\lambda+1-b^2 = 0$

$$\lambda_{1,2} = \frac{-2 \pm\sqrt{4-4+4b^2}}{2} = -1 \pm b :$$

Two negative eigenvalues if $|b|<1$: stable node

A positive and a negative eigenvalse if $|b|>1$: saddle