FALL 95 2nd Exam Solutions

① MISSING.

② ⓐ $\det\begin{bmatrix}2&0&0&-1\\0&2&0&-1\\0&0&2&-1\\-1&-1&-1&2\end{bmatrix} = 2\cdot\det\begin{bmatrix}2&0&-1\\0&2&-1\\-1&-1&2\end{bmatrix} + 1\det\begin{bmatrix}0&2&0\\0&0&2\\-1&-1&-1\end{bmatrix}$

$= 2\cdot[2(3)-1(2)] + 1\cdot[(-1)(4)] = 8-4 = \boxed{4}$

(2b) Subtract the ith row from the $(n+i)$-th row. ⇒ same determinant:

$$\left[\begin{array}{c|c} I & I \\ \hline I & 0 \end{array}\right] \longrightarrow \left[\begin{array}{c|c} I & I \\ \hline 0 & -I \end{array}\right] = \left[\begin{array}{c|c} I & I \\ \hline 0 & -I \end{array}\right]$$ upper triangular.

Det = Prod. of diag. entries.
$= 1^n\cdot(-1)^n = \boxed{(-1)^n}$

③ $\frac{d\vec{x}}{dt} = \begin{bmatrix}5&-3\\9&-7\end{bmatrix}\vec{x} = A\vec{x}$ $\quad \lambda I - A = \begin{bmatrix}\lambda-5&3\\-9&\lambda+7\end{bmatrix}$ $\quad \det = \lambda^2+2\lambda-8$
$= (\lambda+4)(\lambda-2) = 0$

$\lambda_1 = 2 \Rightarrow \left[\begin{array}{cc|c}-3&3&0\\-9&9&0\end{array}\right] \to \left[\begin{array}{cc|c}1&-1&0\\0&0&0\end{array}\right] \Rightarrow \vec{v}_1 = \begin{bmatrix}1\\1\end{bmatrix}$

$\lambda_2 = -4 \Rightarrow \left[\begin{array}{cc|c}-9&3&0\\-9&3&0\end{array}\right] \to \left[\begin{array}{cc|c}1&-1/3&0\\0&0&0\end{array}\right] \Rightarrow \vec{v}_2 = \begin{bmatrix}1\\3\end{bmatrix}$

$P = \begin{bmatrix}1&1\\1&3\end{bmatrix}$ $\quad P^{-1} = \frac{1}{2}\begin{bmatrix}3&-1\\-1&1\end{bmatrix}$

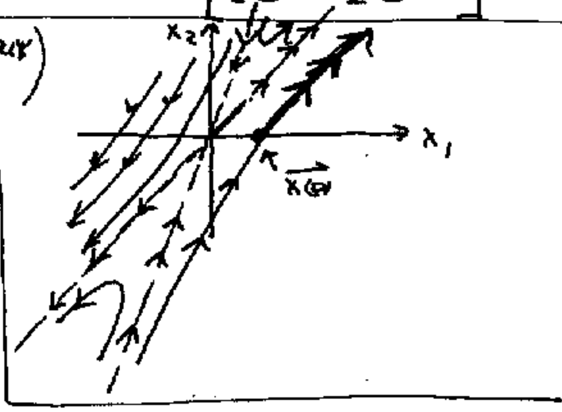
$\vec{x}(0) = \begin{bmatrix}1\\0\end{bmatrix} \to \begin{bmatrix}c_1\\c_2\end{bmatrix} = P^{-1}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}3/2\\-1/2\end{bmatrix}$

Solution: $\vec{x}(t) = [e^{tA}]\vec{x}(0) = Pe^{tD}P^{-1}\vec{x}(0) = \begin{bmatrix}1&1\\1&3\end{bmatrix}\begin{bmatrix}e^{2t}&0\\0&e^{-4t}\end{bmatrix}\begin{bmatrix}3/2\\-1/2\end{bmatrix}$

$P^{-1}AP = D$
$A = PDP^{-1} \quad e^{tA} = Pe^{tD}P^{-1}$ $\quad = \frac{3}{2}e^{2t}\begin{bmatrix}1\\1\end{bmatrix} - \frac{1}{2}e^{-4t}\begin{bmatrix}1\\3\end{bmatrix} = \begin{bmatrix}\frac{3}{2}e^{2t}-\frac{1}{2}e^{-4t}\\ \frac{3}{2}e^{2t}-\frac{3}{2}e^{-4t}\end{bmatrix}$

④ If $\vec{x}(t) = \begin{bmatrix}2^t\\1-2^t\end{bmatrix}$, then $\vec{x}(0) = \begin{bmatrix}1\\0\end{bmatrix}$

(TRAJECTORY FOR 3)



$\vec{x}(1) = \begin{bmatrix}2\\-1\end{bmatrix} = A\vec{x}(0) = A\begin{bmatrix}1\\0\end{bmatrix} \Rightarrow \begin{bmatrix}2\\-1\end{bmatrix}$ is 1st column of A.

If $A = \begin{bmatrix}2&a\\-1&b\end{bmatrix}$, then $A^2 = \begin{bmatrix}2&a\\-1&b\end{bmatrix}\begin{bmatrix}2&a\\-1&b\end{bmatrix} = \begin{bmatrix}4-a&2a+ab\\-2-b&-a+b^2\end{bmatrix}$

$\vec{x}(2) = \begin{bmatrix}4\\-3\end{bmatrix} = A^2\vec{x}(0) = A^2\begin{bmatrix}1\\0\end{bmatrix} \Rightarrow \begin{bmatrix}4-a\\-2-b\end{bmatrix} = \begin{bmatrix}4\\-3\end{bmatrix} \Rightarrow \begin{matrix}a=0\\b=1\end{matrix}$

So $A = \begin{bmatrix}2&0\\-1&1\end{bmatrix}$. You may wish to verify that this really does give the solution stated.

⑤ $q = [x_1\ x_2\ x_3]\begin{bmatrix}6&4&0\\4&6&3\\0&3&6\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \vec{x}^TA\vec{x}$, A symmetric.

CHARACTERISTIC POLYNOMIAL is $f_A(\lambda) = \lambda^3 - 18\lambda^2 + 83\lambda - 66 = (\lambda-11)(\lambda-6)(\lambda-1) = 0$

EIGENVALUES ARE $\{11, 6, 1\}$, so $q(x_1, x_2, x_3)$ is positive definite.

In Rotated coordinates, $q = 11y_1^2 + 6y_2^2 + y_3^2$

(not necessary to find the eigenvectors for this problem.)

MINIMAL VALUE for q on the unit sphere will be 1, where $(y_1, y_2, y_3) = (0,0,1)$ → the direction of the eigenvector for $\lambda_3 = 1$.