HW #12

4.2/ 18, 28, 34

4.3/ 8, 9, 11, 18, 20

18. $Q = \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ \vec{w}_1 & \vec{w}_2 & \vec{w}_3 \\ \downarrow & \downarrow & \downarrow \end{bmatrix}$

refer to 4.2.4 (HW #10) for $\vec{w}_1, \vec{w}_2, \vec{w}_3$

$$Q = \frac{1}{5}\begin{bmatrix} 4 & 3 & 0 \\ 0 & 0 & -5 \\ 3 & -4 & 0 \end{bmatrix}$$

$$R = \begin{bmatrix} \|\vec{v}_1\| & \vec{w}_1 \cdot \vec{v}_2 & \vec{w}_1 \cdot \vec{v}_3 \\ 0 & \|\vec{v}_2 - (\vec{w}_1 \cdot \vec{v}_2)\vec{w}_1\| & \vec{w}_2 \cdot \vec{v}_3 \\ 0 & 0 & \|\vec{v}_3\| \end{bmatrix} = \begin{bmatrix} 5 & 5 & 0 \\ 0 & 35 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

28. from 4.2.14 (HW #10)

$$Q = \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ \vec{w}_1 & \vec{w}_2 & \vec{w}_3 \\ \downarrow & \downarrow & \downarrow \end{bmatrix} = \begin{bmatrix} 1/10 & -1/\sqrt{2} & 0 \\ 7/10 & 0 & 1/\sqrt{2} \\ 1/10 & 1/\sqrt{2} & 0 \\ 7/10 & 0 & -1/\sqrt{2} \end{bmatrix}$$

$$R = \begin{bmatrix} 10 & 10 & 10 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix}$$

34. $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \end{bmatrix}$

$$\Rightarrow \left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\} \text{ is a basis for } \ker(A)$$

HW #12 (pg. 2)

8. a. No. Consider the counter example

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

AA^T is not defined

b. Yes. If A + B are $n \times n$ matrices and ~~AB=~~ $BA = I_n$ then $B = A^{-1}$.

Hence $A^T = A^{-1}$ and $AA^T = AA^{-1} = I_n$

9. Write $A = [\vec{v}_1 \ \vec{v}_2]$. we can express the unit vector $\vec{v}_1$ as $\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$ for some θ. $\vec{v}_2$ must be a unit vector perpendicular to $\vec{v}_1$, so $\vec{v}_2 = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$ or $\vec{v}_2 = \begin{bmatrix} \sin\theta \\ -\cos\theta \end{bmatrix}$

$\therefore$ an orthognal matrix is of the form $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ or $A = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$, representing a rotation or reflection.

11. First, think of the inverse $L = T^{-1}$ of T

$L(x) = A\vec{x} = [\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3] \ \vec{x}$.

we know that $L(\vec{e}_3) = \vec{v}_3 = \begin{bmatrix} 2/3 \\ 2/3 \\ 1/3 \end{bmatrix}$

$\vec{v}_1, \vec{v}_2, \vec{v}_3$ must be an orthonormal basis of $\mathbb{R}^3$

Arbitrarily, we choose $\vec{v}_1 = \begin{bmatrix} -2/3 \\ 1/3 \\ 2/3 \end{bmatrix}$ (this is orthonormal by inspection

Now choose $\vec{v}_2 = \vec{v}_1 \times \vec{v}_3$.

we have $A = \frac{1}{3}\begin{bmatrix} -2 & -1 & 2 \\ 1 & 2 & 2 \\ 2 & -2 & 1 \end{bmatrix}$

Since ~~this~~ A is orthognal the matrix of $T = L^{-1}$ is A^T

$T(\vec{x}) = \frac{1}{3}\begin{bmatrix} -2 & 1 & 2 \\ -1 & 2 & -2 \\ 2 & 2 & 1 \end{bmatrix}\vec{x}$ note: this solution is not unique.

18.a. $a_{ij} = -a_{ji}$

note : diagonal entries must be zero since $a_{ii} = -a_{ii}$

One 3×3 example:

$$\begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & -3 \\ -2 & 3 & 0 \end{bmatrix}$$

b. $(A^2)^T = (AA)^T = A^T A^T = (-A)(-A) = A^2$

so A^2 is symmetric

20. First, use Gram-Schmidt to find the orthonormal basis

$$\vec{w}_1 = \begin{bmatrix} .5 \\ .5 \\ .5 \\ .5 \end{bmatrix} \quad \vec{w}_2 = \begin{bmatrix} -.1 \\ .7 \\ -.7 \\ .1 \end{bmatrix}$$

If $A = [\vec{w}_1 \ \vec{w}_2]$, ~~projection onto W is given~~

the matrix of projection onto W is

$$AA^T = \frac{1}{100}\begin{bmatrix} 26 & 18 & 32 & 24 \\ 18 & 74 & -24 & 32 \\ 32 & -24 & 74 & 18 \\ 24 & 32 & 18 & 26 \end{bmatrix}$$