Solution Set

8.2/8, 10, ~~20~~, 30, 31, 40, 22

8) The zero state is a stable equilibrium solution when all eigenvalues of A are negative. This is the same as saying A is <u>negative definite</u>.

10)
a) $q = x^T A x = [x_1 \; x_2]\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = a_{11}x_1^2 + (a_{21}+a_{12})x_1x_2 + a_{22}x_2^2$

$$\frac{d\vec{x}}{dt} = \begin{bmatrix} d\vec{x}_1/dt \\ d\vec{x}_2/dt \end{bmatrix} = \begin{bmatrix} 2a_{11}x_1 + (a_{21}+a_{12})x_2 \\ (a_{21}+a_{12})x_1 + 2a_{22}x_2 \end{bmatrix} = \begin{bmatrix} 2a_{11} & (a_{21}+a_{12}) \\ (a_{21}+a_{12}) & 2a_{22} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \text{grad}(q)$$

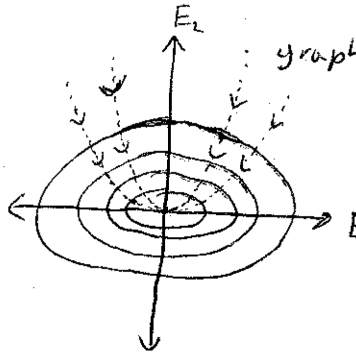
$$\Rightarrow B = \begin{bmatrix} 2a_{11} & (a_{21}+a_{12}) \\ (a_{21}+a_{12}) & 2a_{22} \end{bmatrix}$$

b) notice that B is a symmetric matrix, and, further, notice that $\vec{x}^T(\frac{1}{2}B)\vec{x} = q$. Let S be the orthogonal ~~e-ba~~ matrix consisting of the e-vectors of B. Then $B = SDS^{-1}$ where $D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$. Then $q = \vec{x}^T \frac{1}{2}B\vec{x} = \frac{1}{2}\vec{x}^T(SDS^{-1})x = \frac{1}{2}(\vec{x}^T S)D(S^{-1}\vec{x})$

$$= \frac{1}{2}(S^T\vec{x})^T D(S^{-1}x) = \frac{1}{2}(S^{-1}\vec{x})^T D(S^{-1}\vec{x}) = \frac{1}{2}[c_1 \;\; c_2]\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = (\lambda_1 c_1^2 + \lambda_2 c_2^2)\frac{1}{2}$$

where $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ is the vector $\vec{x}$ expressed w/ respect to the orthonormal e-basis. Finally, note that if q is negative definite, $\lambda_1, \lambda_2 < 0$. So the level curves are solutions to $\lambda_1 c_1^2 + \lambda_2 c_2^2 = 2q$ or $\frac{\lambda_1 c_1^2}{2q} + \frac{\lambda_2 c_2^2}{2q} = 1$ this is graphed on the next pg.

10) b)



: graph of $\frac{\lambda_1 c_1^2}{2q} + \frac{\lambda_2 c_2^2}{2q} = 1$ let $|\lambda_1|<|\lambda_2|$

then looking at the system $\frac{d\vec{x}}{dt} = grad(q) = B\vec{x}$

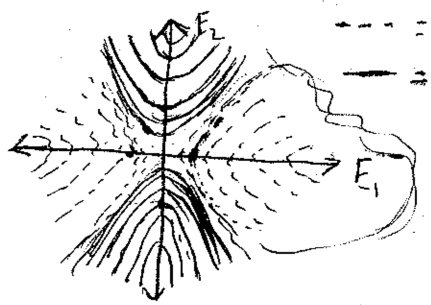
we can graph some trajectories (dashed lines) remembering that $\lambda_2 < \lambda_1 < 0$

sidenote: the gradient of q is in fact $\perp$ to the slope of the level curves of q, as drawn in graph.

$\boxed{\text{The system has a stable equilibrium at } \vec{0}}$

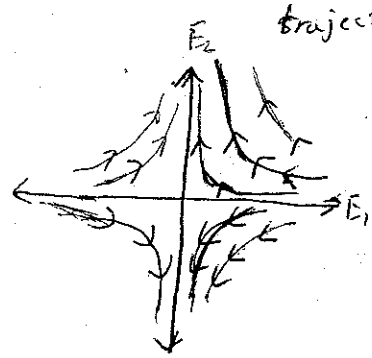
c) indefinite quadratic form means that one e-value of B is posotive, and one is negative $\Rightarrow$ let $\lambda_1 < 0 < \lambda_2$ $\Rightarrow$ $2q = \lambda_1 c_1^2 + \lambda_2 c_2^2$

$\Rightarrow$ $\frac{\lambda_2}{2q} c_2^2 - \frac{|\lambda_1|}{2q} c_1^2 = 1$



- - - = negative q
—— = positive q

trajectories

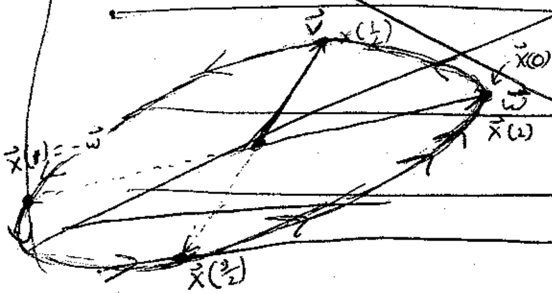


so System does not have a stable equilibrium

d) iff q is negative definite, then the system has a stable equilibrium at $\vec{0}$.

20) $\frac{d\vec{x}}{dt} = A\vec{x} \Rightarrow \vec{x} = \begin{bmatrix}\vec{w} & \vec{v}\end{bmatrix}\begin{bmatrix}\cos\pi t & -\sin\pi t \\ \sin\pi t & \cos\pi t\end{bmatrix}\begin{bmatrix}1 \\ 0\end{bmatrix}$

$= \vec{w}\cos(\pi t) + \vec{v}\sin(\pi t)$



22) at end of Solution set.

30) $\frac{d\vec{x}}{dt} = \begin{bmatrix}7 & 10 \\ -4 & -5\end{bmatrix}\vec{x}$

$\Rightarrow \lambda^2 - 2\lambda + 5 = 0 \Rightarrow \lambda = \frac{2 \pm \sqrt{-16}}{2} = 1 \pm 2i$

$E_+ = \ker\begin{bmatrix}-6+2i & -10 \\ \cdot\cdot & \cdot\cdot\end{bmatrix} = \operatorname{span}\begin{bmatrix}5 \\ -3+i\end{bmatrix} \Rightarrow \vec{w} = \begin{bmatrix}0 \\ 1\end{bmatrix} \quad \vec{v} = \begin{bmatrix}5 \\ -3\end{bmatrix}$

$\vec{x}(0) = \begin{bmatrix}1 \\ 0\end{bmatrix} = \frac{1}{5}\vec{v} + \frac{3}{5}\vec{w} \Rightarrow \vec{x}(t) = e^t\begin{bmatrix}0 & 5 \\ 1 & -3\end{bmatrix}\begin{bmatrix}\cos 2t & -\sin 2t \\ \sin 2t & \cos 2t\end{bmatrix}\begin{bmatrix}3/5 \\ 1/5\end{bmatrix}$

$\vec{x}(t) = \frac{1}{10}e^t\begin{bmatrix}0 & 5 \\ 1 & -3\end{bmatrix}\begin{bmatrix}6\cos 2t - 2\sin 2t \\ 6\sin 2t + 2\cos 2t\end{bmatrix} = \frac{1}{10}e^t\begin{bmatrix}30\sin 2t + 10\cos 2t \\ 20\sin 2t + 0\cos 2t\end{bmatrix}$

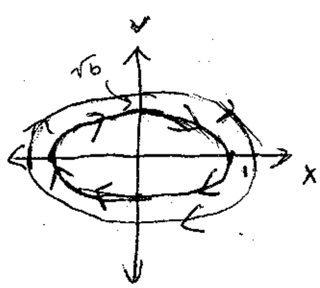
$x_1(t) = (3\sin 2t + \cos 2t)e^t \quad x_2(t) = e^t\, 2\sin 2t$

31)

$\begin{bmatrix}\frac{dx}{dt} \\ \frac{dv}{dt}\end{bmatrix} = \begin{bmatrix}0 & 1 \\ -b & -c\end{bmatrix}\begin{bmatrix}x \\ v\end{bmatrix} \Rightarrow \lambda^2 + c\lambda + b = 0$

$\Rightarrow \lambda = \frac{-c \pm \sqrt{c^2 - 4b}}{2} = -\frac{1}{2}c \pm \frac{1}{2}\sqrt{c^2 - 4b}$

a) <u>if $c = 0$</u> $\lambda = \pm i\sqrt{b} \Rightarrow E_+ = \operatorname{span}\begin{bmatrix}1 \\ i\sqrt{b}\end{bmatrix} \Rightarrow \vec{w} = \begin{bmatrix}0 \\ \sqrt{b}\end{bmatrix} \quad \vec{v} = \begin{bmatrix}1 \\ 0\end{bmatrix}$



the spring oscillates ~~without~~ continuously between some ~~displ~~ displacement $\pm a$ & some velocity $\pm a\sqrt{b}$

31) a) motion is a function of $\cos(\sqrt{b}t)$ & $\sin(\sqrt{b}t)$

$\Rightarrow$ period is $\boxed{T = \frac{2\pi}{\sqrt{b}}}$

b) $\underline{c^2 < 4b}$ $\Rightarrow \lambda = -\frac{1}{2}c \pm \left(\frac{1}{2}\sqrt{4b - c^2}\right)i$

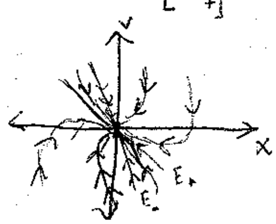
$$E_1 = \begin{bmatrix} 1 \\ -\frac{c}{2} + \frac{\sqrt{4b^2 - c^2}}{2} i \end{bmatrix} \Rightarrow \vec{w} = \begin{bmatrix} 0 \\ \frac{\sqrt{4b^2 - c^2}}{2} \end{bmatrix} \quad \vec{v} = \begin{bmatrix} 1 \\ -\frac{c}{2} \end{bmatrix}$$

The spring oscillates, but the amplitude slowly diminishes toward the equilibrium zero state ($v=0$, $x=0$)

c) $\underline{c^2 > 4b}$ $\lambda = -\frac{1}{2}c \pm \frac{1}{2}\sqrt{c^2 - 4b}$

$\lambda_\pm = -\frac{1}{2}c \pm \sqrt{(\frac{c}{2})^2 - b} \Rightarrow \lambda_- < \lambda_+ < 0$

$$E_+ = \begin{bmatrix} 1 \\ \lambda_+ \end{bmatrix} \quad E_- = \begin{bmatrix} 1 \\ \lambda_- \end{bmatrix}$$



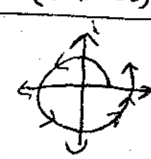
the spring either returns directly to $x=0$ without crossing the equilibrium position, or crosses once, then returns to zero w/o crossing again.

40)

a) $N(t) = 1000(1 + .05i)^t = \boxed{1000\,(1.00125)^t\left(\cos(.05t) + i\sin(.05t)\right)}$

$\boxed{N(1) = 999.999 + i50}$

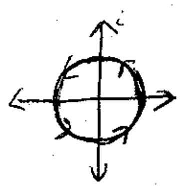
$\boxed{N(2) = 997.5 + i100}$



b) $\boxed{N(t) = 1000e^{.05it}}$

$\boxed{N(1) = 1000e^{.05i}}$

$\boxed{N(2) = 1000e^{.1i}}$



c) annual compounding is better since it ~~improv~~ increases the modulus, while continuous does not.

22) $\begin{bmatrix} 3 & 0 \\ -2.5 & 0.5 \end{bmatrix} \Rightarrow \lambda^2 - 3.5\lambda + 1.5 = 0 \Rightarrow (\lambda-3)(\lambda-.5)$

$\lambda = 3, 0.5$

$\begin{bmatrix} -1.5 & -1 \\ 2 & 0.5 \end{bmatrix} \Rightarrow \lambda^2 + \lambda + 1.25 \Rightarrow \lambda = \frac{-1 \pm \sqrt{-4}}{2} = -\frac{1}{2} \pm i$

a) $\vec{x}(t+1) = \begin{bmatrix} 3 & 0 \\ -2.5 & 0.5 \end{bmatrix} \vec{x}(t)$ $\lambda_3 = 3,\ \lambda = .5 < 1$

$\Rightarrow E_3 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad E_{0.5} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\Rightarrow \boxed{VII}$

b) $\vec{x}(t+1) = \begin{bmatrix} -1.5 & -1 \\ 2 & 0.5 \end{bmatrix} \vec{x}(t) \quad E_+ = \begin{bmatrix} -1 \\ +1+i \end{bmatrix} \Rightarrow \vec{w} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \vec{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$|\lambda| > 1 \Rightarrow \boxed{II}$

c) $\lambda_3 > 0$
$\lambda_{.5} > 0 \Rightarrow \boxed{I}$

d) $Re(\lambda_\pm) < 0 \Rightarrow \boxed{IV}$

e) $\vec{x}' = \begin{bmatrix} -2 & 0 \\ 3 & 1 \end{bmatrix} \vec{x} \Rightarrow \lambda^2 + \lambda - 2 = 0 \quad (\lambda+2)(\lambda-1) = 0$

$\lambda = -2, 1$

$E_{-2} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad E_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow \boxed{V}$