HW #25: Sect. 8.2, #6, 8, 22, 30, 31, 40

Solutions by Dave Freeman

6) $\det \begin{bmatrix} \lambda-3 & 2 \\ -5 & \lambda+3 \end{bmatrix} = (\lambda-3)(\lambda+3)+10 = \lambda^2+1 = 0 \Rightarrow \lambda = \pm i$

eigenvectors:

$\lambda = i$ $\begin{bmatrix} i-3 & 2 \\ -5 & i+3 \end{bmatrix} \to \begin{bmatrix} i-3 & 2 \\ 0 & 0 \end{bmatrix}$ so Kernel = span $\begin{pmatrix} -2 \\ i-3 \end{pmatrix}$

- since eigenvectors come in complex conjugate pairs, the eigenvector for $\lambda = -i$ is $\begin{pmatrix} -2 \\ -i-3 \end{pmatrix}$

The solutions are thus

$$\vec{x}(t) = c_1 e^{it}\begin{pmatrix} -2 \\ -3+i \end{pmatrix} + c_2 e^{-it}\begin{pmatrix} -2 \\ -3-i \end{pmatrix}.$$

If $c_1 = c_2 = 1$, we have

$$\vec{x}(t) = (\cos t + i\sin t)\begin{pmatrix} -2 \\ -3+i \end{pmatrix} + (\cos t - i\sin t)\begin{pmatrix} -2 \\ -3-i \end{pmatrix}$$

$$= \begin{pmatrix} -4\cos t \\ -6\cos t - 2\sin t \end{pmatrix}$$

8) By fact 8.2.4, the zero state is asymptotically stable iff the real parts of all eigenvalues are negative. The eigenvalues of a symmetric matrix A must be real; if they are all negative, then A is <u>negative definite</u>

22)

b) $\det\begin{pmatrix} \lambda+1.5 & 1 \\ -2 & \lambda-0.5 \end{pmatrix} = (\lambda+1.5)(\lambda-0.5)+2 = \lambda^2+\lambda+1.25$

$\lambda = \dfrac{-1 \pm \sqrt{1-5}}{2} = -\frac{1}{2} \pm i$; $|\lambda| = \sqrt{1+\frac{1}{4}} > 1$

$\Rightarrow$ solutions spiral away from the origin

$\Rightarrow$ portrait . II.

a) $\lambda = 3,\ 0.5 \Rightarrow$ one eigenvector goes towards the origin, the other goes away

eigenvectors: $\lambda = 0.5$ $\begin{bmatrix} 2.5 & 0 \\ 2.5 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is eigenvector going toward the origin $\Rightarrow$ portrait VII

c) $\lambda = 3,\ 0.5 \Rightarrow$ both eigenvectors go away from the origin $\Rightarrow$ portrait I.

d) $\lambda = -\frac{1}{2} \pm i$; $Re(\lambda) < 0$, so solutions spiral in towards the origin $\Rightarrow$ portrait IV

e) $\lambda = -2, 1 \Rightarrow$ one eigenvector goes towards the origin, the other goes away

$\lambda = 1$ $\begin{bmatrix} -3 & 0 \\ 3 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is the eigenvector going away from the origin $\Rightarrow$ portrait V.

30) $f_A(\lambda) = (\lambda - 7)(\lambda + 5) + 40 = \lambda^2 - 2\lambda + 5$

$\lambda = \dfrac{2 \pm \sqrt{4-20}}{2} = 1 \pm 2i$

$E_1 = \ker \begin{bmatrix} -6+2i & -10 \\ 4 & 6+2i \end{bmatrix} = \text{span} \begin{pmatrix} 10 \\ -6+2i \end{pmatrix} = \text{span} \begin{pmatrix} 5 \\ -3+i \end{pmatrix} = \begin{pmatrix} 5 \\ -3 \end{pmatrix} + i \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

by fact 8.2.6,

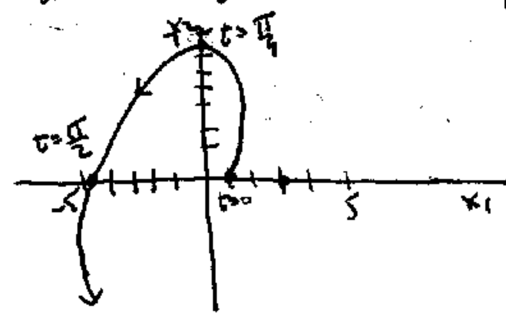
$\vec{x}(t) = e^t \begin{bmatrix} 0 & 5 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} \cos 2t & -\sin 2t \\ \sin 2t & \cos 2t \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$

$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = a \begin{pmatrix} 0 \\ 1 \end{pmatrix} + b \begin{pmatrix} 5 \\ -3 \end{pmatrix} \Rightarrow b = .2,\ a = .6$

$\vec{x}(t) = e^t \begin{bmatrix} 0 & 5 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} \cos 2t & -\sin 2t \\ \sin 2t & \cos 2t \end{bmatrix} \begin{bmatrix} .6 \\ .2 \end{bmatrix}$

$= e^t \begin{bmatrix} 5\sin 2t & 5\cos 2t \\ \cos 2t - 3\sin 2t & -\sin 2t - 3\cos 2t \end{bmatrix} \begin{bmatrix} .6 \\ .2 \end{bmatrix} = e^t \begin{bmatrix} 3\sin 2t + \cos 2t \\ -2\sin 2t \end{bmatrix}$

sketch:



31) $f_A(\lambda) = \lambda(\lambda + c) + b = \lambda^2 + c\lambda + b \Rightarrow \lambda = \dfrac{-c \pm \sqrt{c^2 - 4b}}{2}$

a) $c = 0 \Rightarrow \lambda = \pm i\sqrt{b}$

eigenvectors are $\begin{bmatrix} 1 \\ \pm i\sqrt{b} \end{bmatrix}$

so $\begin{pmatrix} x \\ v \end{pmatrix} = c_1 e^{i\sqrt{b}t} \begin{pmatrix} 1 \\ i\sqrt{b} \end{pmatrix} + c_2 e^{-i\sqrt{b}t} \begin{pmatrix} 1 \\ -i\sqrt{b} \end{pmatrix}$

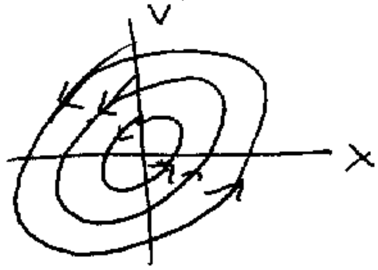
$=$ ~~[illegible]~~

or by fact 8.7.6

$\begin{pmatrix} x \\ v \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ \sqrt{b} & 0 \end{bmatrix} \begin{bmatrix} \cos\sqrt{b}t & -\sin\sqrt{b}t \\ \sin\sqrt{b}t & \cos\sqrt{b}t \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$

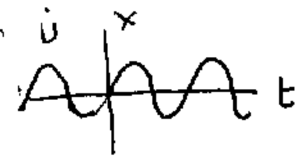
$= \begin{bmatrix} \sin\sqrt{b}t & \cos\sqrt{b}t \\ \sqrt{b}\cos\sqrt{b}t & -\sqrt{b}\sin\sqrt{b}t \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a\sin\sqrt{b}t + b\cos\sqrt{b}t \\ a\sqrt{b}\cos\sqrt{b}t - b\sqrt{b}\sin\sqrt{b}t \end{bmatrix}$

The trajectories are ellipses:



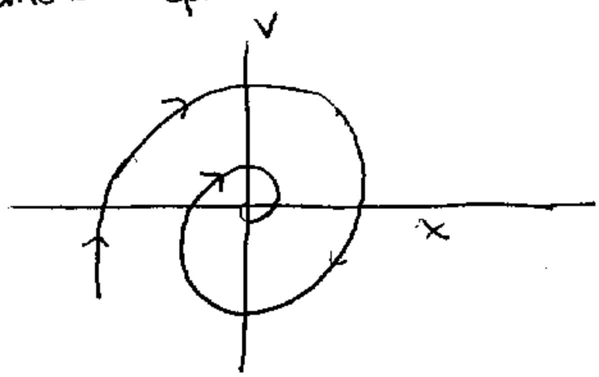
The zero state is not stable, since $Re(\lambda) \geq 0$.

The motion of the block is the x-component, which is simple harmonic motion of period $\dfrac{2\pi}{\sqrt{b}}$.

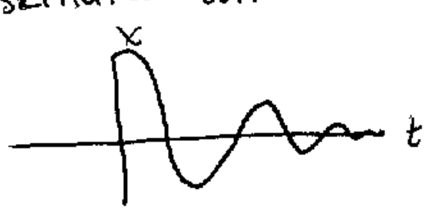


b) $c^2 < 4b$ $\quad \lambda = -\frac{c}{2} \pm i\sqrt{4b - c^2}$

Solutions spiral in towards the origin:

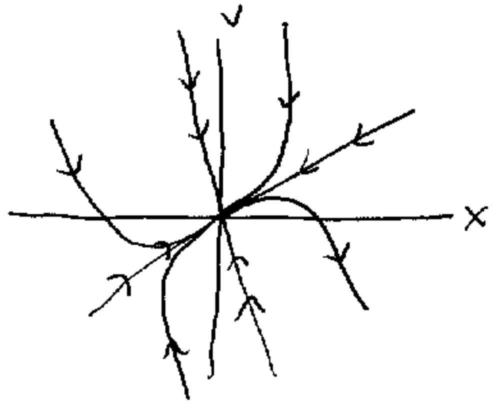


The block oscillates with decreasing amplitude:

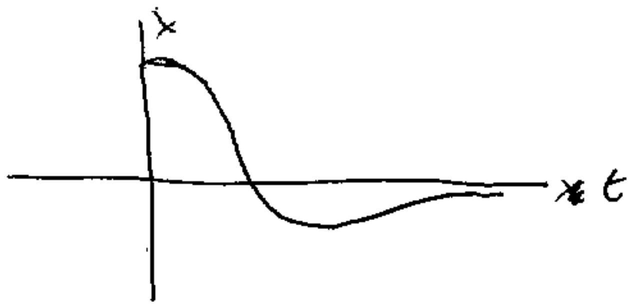


e) $c^2 > 4b \Rightarrow \lambda = -\frac{c}{2} \pm \frac{\sqrt{c^2-4b}}{2}$, which is real and negative

$\Rightarrow$ solutions move towards the origin



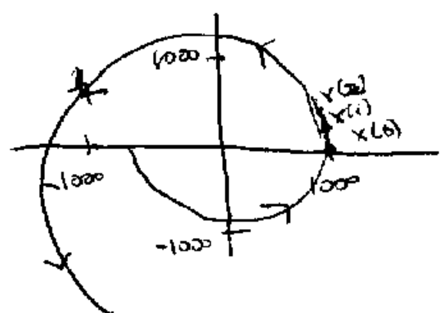
In terms of the motion of the block, the block crosses the equilibrium point at most once:



40) a) $x(t) = 1000(1+.05i)^t$

$x(1) = 1000 + 50i$

$x(2) = 1000(1+.05i)^2 = 997.5 + 100i$

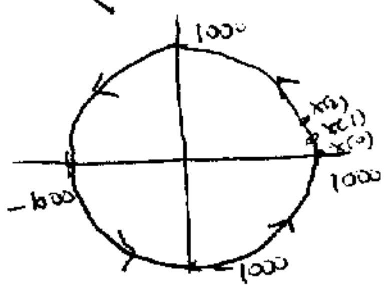


b) $x(t) = 1000\, e^{.05it}$

$= 1000(\cos .05t + i \sin .05t)$

$x(1) = 998.8 + 49.97i$

$x(2) = 995.0 + 99.83i$



c) annual compounding is better.

The modulus of the continuously compounded account remains fixed, while that of the annually compounded account grows like 1000×1.05^t.