Math 21b Solution Set Section 9.1

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Question 65

General Comments

Recall that to verify that a subset of C^∞ is a subspace, we need to verify that a subset has the following properties:

- 0 is contained in the subset
- the subset is closed under vector addition
- the subset is closed under scalar multiplication

For C^∞, these properties are dependent on basic rules of calculus that are most efficiently proved in other textbooks and that I therefore only allude or refer to.

Part (i)

This is NOT a subspace

Let $f = |x|$. Then f is continuous but not C^∞ since f is not differentiable at 0, and thus f can not be in a subset of the space C^∞.

Part (ii)

This IS a subspace.

0 is clearly in this subset, and if f and g are in this subset, then $(\mathrm{f}+\mathrm{g})(0) + (\mathrm{f}+\mathrm{g})'(0) = (\mathrm{f})(0) + (\mathrm{f})'(0) + (\mathrm{g})(0) + (\mathrm{g})'(0) = 0$, so the subset is closed under addition. Finally, if f is in this subset, then if k is a scalar we observe $k \cdot \mathrm{f}(0) + k \cdot \mathrm{f}'(0) = k \cdot 0 = 0$, so the subset is closed under scalar multplication.

Part (iii)

This IS a subspace.

0 is clearly in this subset, and if f and g are in this subset, then $(\mathrm{f}+\mathrm{g}) + (\mathrm{f}+\mathrm{g})' = (\mathrm{f}) + (\mathrm{f})' + (\mathrm{g}) + (\mathrm{g})' = 0$, so the subset is closed under addition. Finally, if f is in this subset, then if k is a scalar we observe $k \cdot \mathrm{f} + k \cdot \mathrm{f}' = k \cdot 0 = 0$, so the subset is closed under scalar multplication.

Part (iv)

This is NOT a subspace.

0 is clearly not in this subset

Question 66

Part (i)

by example 2 on page 3 of the Section 9-1 Handout we see that the set IS linearly independent.

Part (ii)

The set is NOT linearly independent. $(1 + t) + t^2 = 1 + t + t^2$

Part (iii)

The set IS linearly independent

since $|\sin(t)| \leq 1$ and $\exp(t)$, $\exp(-t)$ are always positive and are unbounded, any linear dependence relation is impossible. Similarly, $k_1 \exp(t) + k_2 \exp(-t) = 0 \Rightarrow \exp(2t) = \text{ a constant }$.

Part (iv)

The set is NOT linearly independent

By the addition law for the sine function in trigonometry:

$$\sin(A + B) = \sin A \cos B + \cos A \sin B \Rightarrow \sin(t + \frac{\pi}{3}) = \sin t \cos(\frac{\pi}{3}) + \cos t \sin(\frac{\pi}{3})$$

Question 67

Part (i)

LINEAR

T is linear since

$$\mathrm{T}(f + g) = (\mathrm{f} + \mathrm{g})(0) = \mathrm{f}(0) + \mathrm{g}(0) = \mathrm{T}(f) + \mathrm{T}(g)$$

and

$$\mathrm{T}(k \cdot f) = k \cdot \mathrm{f}(0) = k\,\mathrm{T}(f)$$

Part (ii)

NOT LINEAR

$$k \cdot \mathrm{T}(f) \neq k^2 f^2 + k f' = \mathrm{T}(kf)$$

Part (iii)

LINEAR Using basic properties of matrices, we see that

$$\mathrm{T}(f+g) = \begin{bmatrix} (\mathrm{f}+\mathrm{g})(0) \\ (\mathrm{f}+\mathrm{g})(1) \end{bmatrix} = \begin{bmatrix} \mathrm{f}(0) \\ \mathrm{f}(1) \end{bmatrix} + \begin{bmatrix} \mathrm{g}(0) \\ \mathrm{g}(1) \end{bmatrix} = \mathrm{T}(f) + \mathrm{T}(g)$$

and similarly. T commutes with scalar multplication by Fact 1.3.7. of Brestcher's book.

Part (iv)

LINEAR

Definite integrals are linear from their basic properties from calculus.

Question 68

$\mathrm{f}''(t) - \mathrm{f}(0) = 0 \Rightarrow \mathrm{f}^{prime}(t) = [\mathrm{f}(0)]t + b \Rightarrow \mathrm{f}(t) = (1/2)[\mathrm{f}(0)]t^2 + bt + c = (1/2)ct^2 + bt + c$

where b and c are arbitrary constants. A basis for this 2 dimensional set of functions(one of many possible bases) is

$$\{(1 + (1/2)t^2), t\}$$

Question 69

Given arbitrary values of a and b for $\mathrm{f}(0)$, $f'(0)$, we can find a function in C^∞ that has those values by picking $\mathrm{f}(t) = a + bt$. Therefore, the image of T is the set of polynomials $a + bt + [a+b]t^2$ where a,b are arbitrary. One basis for this 2 dimensional subspace of C^∞ is $\{(1+t^2), (t+t^2)\}$.

Question 54

$\mathrm{f} \in \ker(T - (\lambda I)) \Rightarrow (1-\lambda)f + f' = 0$. Thus, $\mathrm{f}' = (\lambda - 1)(f) \Rightarrow \mathrm{f}(t) = k \cdot \exp((1-\lambda)t)$. So for any arbritray λ, λ is an eigenvalue with an eigenvector $\exp((1-\lambda)t)$.