Math 21b Solutions HW#4

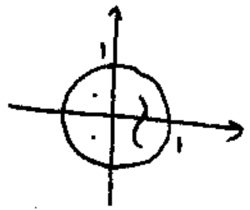
2.1

5) $A = \begin{bmatrix} | & | & | \\ T(e_1) & T(e_2) & T(e_3) \\ | & | & | \end{bmatrix} = \boxed{\begin{bmatrix} 7 & 6 & -13 \\ 11 & 9 & 17 \end{bmatrix}}$

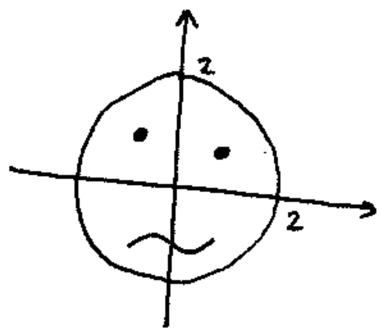
8) $\begin{array}{c} \begin{array}{cccc} x_1 & x_2 & y_1 & y_2 \end{array} \\ \left(\begin{array}{cc|cc} 1 & 7 & 1 & 0 \\ 3 & 20 & 0 & 1 \end{array}\right) \end{array} {\scriptstyle -3\,I} \longrightarrow \left(\begin{array}{cc|cc} 1 & 7 & 1 & 0 \\ 0 & -1 & -3 & 1 \end{array}\right) {\scriptstyle \times(-1)} \longrightarrow \left(\begin{array}{cc|cc} 1 & 7 & 1 & 0 \\ 0 & 1 & 3 & -1 \end{array}\right) {\scriptstyle -7\,II}$

$\left(\begin{array}{cc|cc} 1 & 0 & -20 & 7 \\ 0 & 1 & 3 & -1 \end{array}\right) \quad \therefore \quad \boxed{\begin{aligned} x_1 &= -20y_1 + 7y_2 \\ x_2 &= 3y_1 - y_2 \end{aligned}}$

24)



25)



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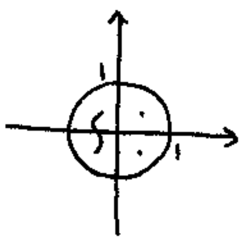
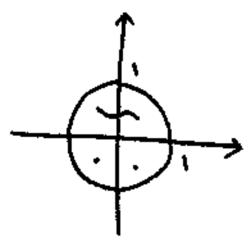


27)



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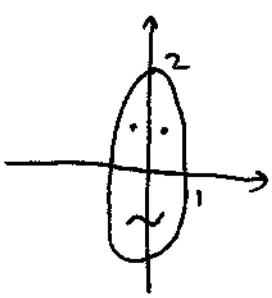

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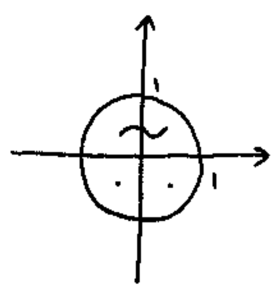
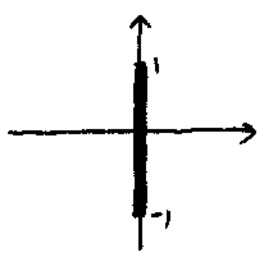
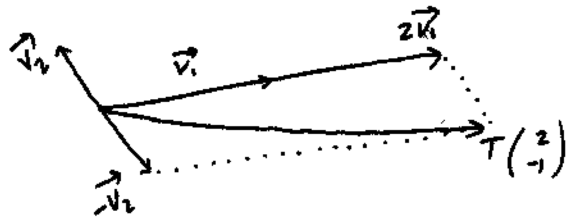


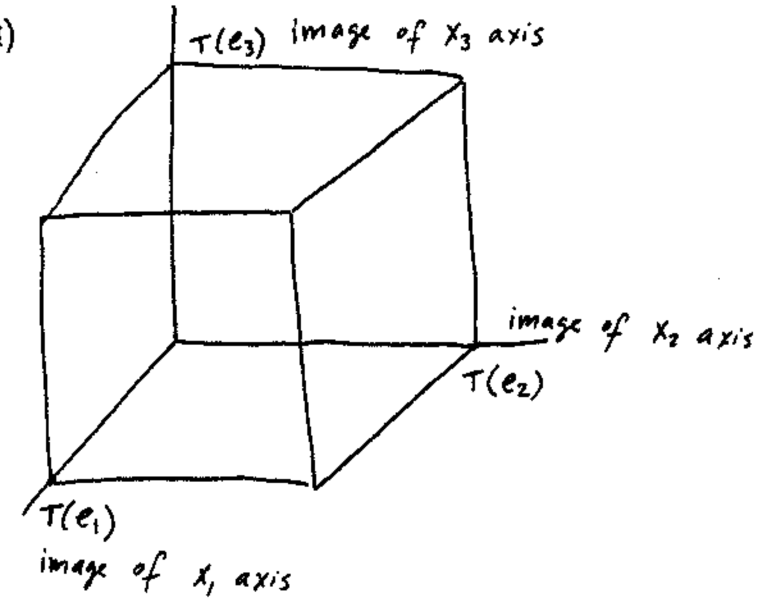
30)



38) $T\begin{pmatrix}2\\-1\end{pmatrix} = \begin{pmatrix}\vec{v}_1 & \vec{v}_2\end{pmatrix}\begin{pmatrix}2\\-1\end{pmatrix} = 2\vec{v}_1 - \vec{v}_2$



42) a)



b) We need to solve $\begin{pmatrix}-1/2 & 1 & 0\\ -1/2 & 0 & 1\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$

$$\left(\begin{array}{ccc|c}-1/2 & 1 & 0 & 0\\ -1/2 & 0 & 1 & 0\end{array}\right) \times(-2) \longrightarrow \left(\begin{array}{ccc|c}1 & -2 & 0 & 0\\ 1 & 0 & -2 & 0\end{array}\right)_{-\mathrm{I}} \longrightarrow \left(\begin{array}{ccc|c}1 & -2 & 0 & 0\\ 0 & 2 & -2 & 0\end{array}\right)_{/2}$$

$$\left(\begin{array}{ccc|c}1 & -2 & 0 & 0\\ 0 & 1 & -1 & 0\end{array}\right)^{+2\mathrm{II}} \longrightarrow \left(\begin{array}{ccc|c}1 & 0 & -2 & 0\\ 0 & 1 & -1 & 0\end{array}\right) \quad \therefore \quad \begin{array}{l} x_1 - 2x_3 = 0\\ x_2 - x_3 = 0\end{array}$$

Let $x_3 = t$, then $x_1 = 2t$, $x_2 = t$.

$$\boxed{\text{Thus, all pts. of the form } \begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{pmatrix}2\\1\\1\end{pmatrix} t}$$

43) a) We need to check whether $T(a\vec{x}+\vec{y}) = aT(\vec{x}) + T(\vec{y})$.

Given $a, \vec{x}, \vec{y}$; $T(a\vec{x}+\vec{y}) = \vec{v}\cdot(a\vec{x}+\vec{y}) = \sum_{i=1}^{3} v_i(ax_i + y_i)$

$= \sum_{i=1}^{3} a v_i x_i + v_i y_i = a\sum_{i=1}^{3} v_i x_i + \sum_{i=1}^{3} v_i y_i = a\vec{v}\cdot\vec{x} + \vec{v}\cdot y$

$= aT(\vec{x}) + T(\vec{y})$. $\boxed{\therefore T \text{ is linear}}$

Now, to find the matrix of T, consider $T(\vec{x}) = \begin{pmatrix}2\\3\\4\end{pmatrix}\cdot\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$

$= 2x_1 + 3x_2 + 4x_3 = \begin{bmatrix}2 & 3 & 4\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$. $\boxed{\therefore \begin{bmatrix}2 & 3 & 4\end{bmatrix} \text{ is the matrix of } T}$

b) As we've shown in a) T is linear for any $\vec{v}$, and it's easy to see that the matrix of $T(\vec{x}) = \begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix}\cdot\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$ is $\boxed{\begin{bmatrix}v_1 & v_2 & v_3\end{bmatrix}}$.

c) Given linear $T: \mathbb{R}^3 \to \mathbb{R}$, let $[a \ b \ c]$ be its matrix.

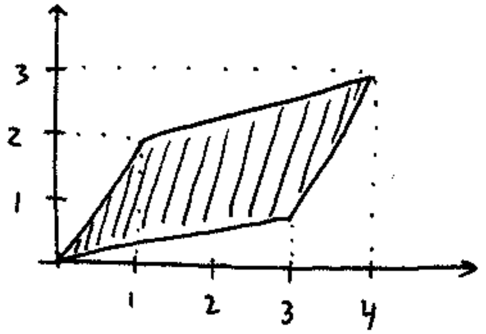
Then $T\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = [a \ b \ c]\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = ax_1 + bx_2 + cx_3 = \begin{pmatrix}a\\b\\c\end{pmatrix}\cdot\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$

$\boxed{\therefore \vec{v} = \begin{pmatrix}a\\b\\c\end{pmatrix}}$

Math 21b Solutions HW #5

2.2

1) $T\begin{pmatrix}0\\0\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$, $T\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}3\\1\end{pmatrix}$, $T\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}4\\3\end{pmatrix}$, $T\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}1\\2\end{pmatrix}$



6) Let $\vec{u}$ be unit vector on L. $\vec{u}=\frac{1}{\sqrt{2^2+1^2+2^2}}\begin{pmatrix}2\\1\\2\end{pmatrix}=\frac{1}{3}\begin{pmatrix}2\\1\\2\end{pmatrix}$

Then, $\text{Proj}_L\begin{pmatrix}1\\1\\1\end{pmatrix}=\left[\vec{u}\cdot\begin{pmatrix}1\\1\\1\end{pmatrix}\right]\vec{u}=\frac{5}{3}\cdot\vec{u}=\frac{5}{9}\begin{pmatrix}2\\1\\2\end{pmatrix}=\boxed{\begin{pmatrix}10/9\\5/9\\10/9\end{pmatrix}}$

7) By fact 2.2.7 $\text{Ref}_L\begin{pmatrix}1\\1\\1\end{pmatrix}=2\left(\vec{u}\cdot\begin{pmatrix}1\\1\\1\end{pmatrix}\right)\vec{u}-\begin{pmatrix}1\\1\\1\end{pmatrix}$, w/ $\vec{u}$ same as in problem 6. $\therefore$ $\text{Ref}_L\begin{pmatrix}1\\1\\1\end{pmatrix}=2\cdot\frac{5}{3}\cdot\frac{1}{3}\begin{pmatrix}2\\1\\2\end{pmatrix}-\begin{pmatrix}1\\1\\1\end{pmatrix}=\boxed{\begin{pmatrix}11/9\\1/9\\11/9\end{pmatrix}}$

10) Here, $\vec{u}=\frac{1}{\sqrt{4^2+3^2}}\begin{pmatrix}4\\3\end{pmatrix}=\begin{pmatrix}0.8\\0.6\end{pmatrix}$.

So, $T(\vec{x})=A\vec{x}=(\vec{u}\cdot\vec{x})\vec{u}=(0.8x_1+0.6x_2)\begin{pmatrix}0.8\\0.6\end{pmatrix}=\begin{pmatrix}0.64x_1+0.48x_2\\0.48x_1+0.36x_2\end{pmatrix}$

$=\begin{pmatrix}0.64&0.48\\0.48&0.36\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}$. $\therefore$ $\boxed{A=\begin{pmatrix}0.64&0.48\\0.48&0.36\end{pmatrix}}$

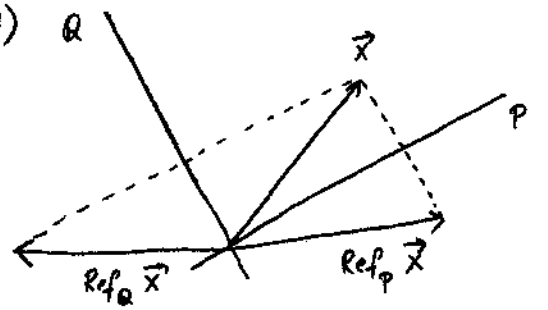
14) a) Generalizing the procedure of problem 10, we see that A is the matrix whose ijth entry $\boxed{A_{ij}=u_i u_j}$.

b) Sum of diagonal entries $\sum_{i=j} A_{ij}=\sum_{i=1}^{n} u_i^2=\boxed{1}$ as $\vec{u}$ is a unit vector.

2.2

20) $T(e_1) = e_1$, $T(e_2) = -e_2$, $T(e_3) = e_3$; $\therefore \boxed{A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}$

41)



$\therefore \boxed{Ref_Q \vec{x} = -Ref_P \vec{x}}$

47) Let $T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

a) $f(t) = \begin{pmatrix} a\cos t + b \sin t \\ c\cos t + d \sin t \end{pmatrix} \begin{pmatrix} -a \sin t + b\cos t \\ -c \sin t + d \cos t \end{pmatrix}$

$= (a\cos t + b \sin t)(-a \sin t + b \cos t) + (c\cos t + d \sin t)(-c \sin t + d\cos t)$

$\therefore f$ is cont, since cos, sin, and constant functions are cont., and sums and products of cont. functions are cont.

b) $f(\pi/2) = T\begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot T\begin{pmatrix} -1 \\ 0 \end{pmatrix} = -\left(T\begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot T\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)$ as T is linear

$f(0) = T\begin{pmatrix} 1 \\ 0 \end{pmatrix} T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = T\begin{pmatrix} 0 \\ 1 \end{pmatrix} T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = -f(\pi/2)$.

c) by b) either $f(\pi/2) = f(0) = 0$ OR $f(0)$ & $f(\pi/2)$ have different signs, & therefore zero lies b/w $f(0)$ & $f(\pi/2)$. As f is cont. by part a), intermediate value Thm. tells us there is a number c b/w 0 & $\pi/2$ s.t. $f(c) = 0$.

d) Note that $\begin{bmatrix} \cos t \\ \sin t \end{bmatrix}$ and $\begin{bmatrix} -\sin t \\ \cos t \end{bmatrix}$ are perpendicular unit vectors for any t.

Let $\vec{v}_1 = \begin{pmatrix} \cos(c) \\ \sin(c) \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} -\sin(c) \\ \cos(c) \end{pmatrix}$, using number c from part c), then

$f(c) = T(\vec{v}_1) \cdot T(\vec{v}_2) = 0 \Rightarrow T(\vec{v}_1) \perp T(\vec{v}_2)$.

Math 21 b solutions HW #6

2.3

10) $$\left(\begin{array}{ccc|ccc} 1&1&1&1&0&0\\ 1&2&3&0&1&0\\ 1&3&6&0&0&1 \end{array}\right)\begin{array}{l} \\ -\mathrm{I}\\ -\mathrm{I}\end{array} \to \left(\begin{array}{ccc|ccc} 1&1&1&1&0&0\\ 0&1&2&-1&1&0\\ 0&2&5&-1&0&1 \end{array}\right)\begin{array}{l} -\mathrm{II}\\ \\ -2\mathrm{II}\end{array} \to \left(\begin{array}{ccc|ccc} 1&0&-1&2&-1&0\\ 0&1&2&-1&1&0\\ 0&0&1&1&-2&1 \end{array}\right)\begin{array}{l} +\mathrm{III}\\ -2\mathrm{III}\\ \\ \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1&0&0&3&-3&1\\ 0&1&0&-3&5&-2\\ 0&0&1&1&-2&1 \end{array}\right) \quad \therefore \boxed{\begin{pmatrix} 1&1&1\\ 1&2&3\\ 1&3&6 \end{pmatrix}^{-1} = \begin{pmatrix} 3&-3&1\\ -3&5&-2\\ 1&-2&1 \end{pmatrix}}$$

20) $$\left(\begin{array}{ccc|ccc} 1&3&3&1&0&0\\ 1&4&8&0&1&0\\ 2&7&12&0&0&1 \end{array}\right)\begin{array}{l} \\ -\mathrm{I}\\ -2\mathrm{I}\end{array} \to \left(\begin{array}{ccc|ccc} 1&3&3&1&0&0\\ 0&1&5&-1&1&0\\ 0&1&6&-2&0&1 \end{array}\right)\begin{array}{l} -3\mathrm{II}\\ \\ -\mathrm{II}\end{array}$$

$$\left(\begin{array}{ccc|ccc} 1&0&-12&4&-3&0\\ 0&1&5&-1&1&0\\ 0&0&1&-1&-1&1 \end{array}\right)\begin{array}{l} +12\mathrm{III}\\ -5\mathrm{III}\\ \\ \end{array} \to \left(\begin{array}{ccc|ccc} 1&0&0&-8&-15&12\\ 0&1&0&4&6&-5\\ 0&0&1&-1&-1&1 \end{array}\right)$$

$$\therefore \boxed{\begin{aligned} x_1 &= -8y_1 - 15y_2 + 12y_3\\ x_2 &= 4y_1 + 6y_2 - 5y_3\\ x_3 &= -y_1 - y_2 + y_3 \end{aligned}}$$

26) Invertible. Consider $\boxed{\begin{pmatrix} x_1\\ x_2 \end{pmatrix} = \begin{pmatrix} \sqrt[3]{y_2 - y_1}\\ y_1 \end{pmatrix}}$

30) $$\mathrm{Rref}\begin{pmatrix} 0&1&b\\ -1&0&c\\ -b&-c&0 \end{pmatrix} = \begin{pmatrix} 1&0&-c\\ 0&1&b\\ 0&0&0 \end{pmatrix}$$ $\therefore$ **The Matrix is not invertible regardless of the values of b & c.**

40) If you apply an elementary row operation to a matrix w/ 2 equal columns, the resulting matrix will also have 2 equal columns.
$\therefore$ Rref(A) has 2 equal columns. $\therefore$ $\mathrm{Rref}(A) \neq I_n$.
$\therefore$ A is not invertible.

41) a) Invertible: the transformation is its own inverse.

b) Not invertible: It is neither true that the image of the transformation is all of $\mathbb{R}^3$ (it is in fact $\mathbb{R}^2$), nor that the transformation is one-to-one (2 different vectors can have the same projection onto a given plane).

c) Invertible: The inverse is a dilation by $1/5$.

d) Invertible: The inverse is a rotation about the same axis through the same angle in the opposite direction.

42) Permutation matrices are invertible as they can be row-reduced to I_n by simple row swaps. The inverse of a permutation matrix is also a permutation matrix since $[I_n \mid A^{-1}]$ can be obtained from $[A \mid I_n]$ by simple row swaps.