Math 21b
Homework #7 (TTh: HW #5)
Solutions by Dave Freeman

Sect. 2.4

14) The matrix product AB is defined when the number of columns of A is equal to the number of rows of B.

A: 2×2 B: 1×3 C: 3×3 D: 3×1 E: 1×1

$$AA = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$$

$$BC = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1+4+9 & 0+2+6 & -1+0+2 \end{bmatrix} = \begin{bmatrix} 14 & 8 & 2 \end{bmatrix}$$

$$BD = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1+2+3 \end{bmatrix} = \begin{bmatrix} 6 \end{bmatrix}$$

$$CC = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1+0-3 & 0+0-2 & -1+0-1 \\ 2+2+0 & 0+1+0 & -2+0+0 \\ 3+4+3 & 0+2+2 & -3+0+1 \end{bmatrix} = \begin{bmatrix} -2 & -2 & -2 \\ 4 & 1 & -2 \\ 10 & 4 & -2 \end{bmatrix}$$

$$CD = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1+0-1 \\ 2+1+0 \\ 3+2+1 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 6 \end{bmatrix}$$

$$DB = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix} \qquad DE = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\begin{bmatrix} 5 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \\ 5 \end{bmatrix}$$

$$EB = \begin{bmatrix} 5 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 5 & 10 & 15 \end{bmatrix} \qquad EE = \begin{bmatrix} 5 \end{bmatrix}\begin{bmatrix} 5 \end{bmatrix} = \begin{bmatrix} 25 \end{bmatrix}$$

These 9 are the only products that are defined.

17) $(A+B)^2 = (A+B)(A+B) = A\cdot A + AB + BA + B\cdot B = A^2 + BA + AB + B^2$

Not necessarily true.

e.g. $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

$$(A+B)^2 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 4+1 & 2+2 \\ 2+2 & 4+1 \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}$$

$$A^2+2AB+B^2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} + 2\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} + 2\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 4 \\ 4 & 4 \end{bmatrix}$$

19) Not necessarily true.

For example, $A=\begin{bmatrix}1&0\\0&1\end{bmatrix}$ $B=\begin{bmatrix}-1&0\\0&-1\end{bmatrix}$. Both are invertible, but

$A+B=\begin{bmatrix}0&0\\0&0\end{bmatrix}$ is not invertible.

In addition, $C=\begin{bmatrix}1&0\\0&1\end{bmatrix}$ A and C are their own inverses,

but $A+C=\begin{bmatrix}2&0\\0&2\end{bmatrix}$ has inverse $\begin{bmatrix}\frac{1}{2}&0\\0&\frac{1}{2}\end{bmatrix} \neq A^{-1}+C^{-1}$

22) Not necessarily true: $ABA^{-1}=B \Rightarrow (ABA^{-1})(A)=(B)(A)$

$\Rightarrow (AB)(A^{-1}A)=(BA) \Rightarrow AB=BA$

So the identity is true if and only if $AB=BA$.

Let $A=\begin{bmatrix}1&1\\0&1\end{bmatrix}$ $B=\begin{bmatrix}1&0\\1&1\end{bmatrix}$

$A^{-1}=\begin{bmatrix}1&-1\\0&1\end{bmatrix}$ $B^{-1}=\begin{bmatrix}1&0\\-1&1\end{bmatrix}$

$$ABA^{-1}=\left(\begin{bmatrix}1&1\\0&1\end{bmatrix}\begin{bmatrix}1&0\\1&1\end{bmatrix}\right)\begin{bmatrix}1&-1\\0&1\end{bmatrix}=\begin{bmatrix}2&1\\1&1\end{bmatrix}\begin{bmatrix}1&-1\\0&1\end{bmatrix}=\begin{bmatrix}2&-1\\1&0\end{bmatrix}\neq B$$

29) Any matrix of the form $\begin{bmatrix}3a&3b\\-a&-b\end{bmatrix}$ will work:

$$\begin{bmatrix}1&3\\2&6\end{bmatrix}\begin{bmatrix}3a&3b\\-a&-b\end{bmatrix}=\begin{bmatrix}3a-3a&3b-3b\\6a-6a&6b-6b\end{bmatrix}=\begin{bmatrix}0&0\\0&0\end{bmatrix}.$$

35) a) $\vec{x}=\vec{0}$ is the only solution. Suppose there were some nonzero $\vec{x}$ for which $A\vec{x}=\vec{0}$. Then $B(A\vec{x})=(BA)\vec{x}=\vec{0}\neq\vec{x}$, contradicting the fact that BA is the identity.

b) The system $B\vec{x}=\vec{b}$ is satisfied by $\vec{x}=A\vec{b}$, since $B(A\vec{b})=\vec{b}$ for all $\vec{b}$ in $\mathbb{R}^m$.

Thus the system is consistent for all $\vec{b}$ in $\mathbb{R}^m$.

c) From (a), $A\vec{x}=\vec{0}$ has a unique solution, so there is a 1 in each column ~~of A~~ rref A, so rank $A=m$

From (b), $B\vec{x}=\vec{b}$ has a solution for all $\vec{b}$, so there is a 1 in each row of B, so rank $B=m$

d) If $n<m$, then there could not be m leading 1's in rref A, contradicting (a).

43) Let A represent rotation by $\frac{2\pi}{3}$ radians. Then A^3 is rotation by 2π, which is the identity.

Thus $A = \begin{bmatrix} \cos\frac{2\pi}{3} & -\sin\frac{2\pi}{3} \\ \sin\frac{2\pi}{3} & \cos\frac{2\pi}{3} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$ is a non-identity matrix such that $A^3 = I_2$

(rotation by $\frac{4\pi}{3}$ also works).

44) Solution #1

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$ gives

$$\begin{aligned} a+2b &= 2 \\ 2a+5b &= 1 \\ c+2d &= 1 \\ 2c+5d &= 3 \end{aligned} \qquad \text{rref}\left[\begin{array}{cccc|c} 1 & 2 & 0 & 0 & 2 \\ 2 & 5 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 2 & 5 & 3 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & & & & 8 \\ & 1 & & & -3 \\ & & 1 & & -1 \\ & & & 1 & 1 \end{array}\right] \text{ so } A = \begin{bmatrix} 8 & -3 \\ -1 & 1 \end{bmatrix}$$

Solution #2

Find the inverse of $\begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$:

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 2 & 5 & 0 & 1 \end{array}\right]_{-2R_1} \longrightarrow \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{array}\right]^{-2R_2} \longrightarrow \left[\begin{array}{cc|cc} 1 & 0 & 5 & -2 \\ 0 & 1 & -2 & 1 \end{array}\right]$$

so $A = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 10-2 & -4+1 \\ 5-6 & -2+3 \end{bmatrix} = \begin{bmatrix} 8 & -3 \\ -1 & 1 \end{bmatrix}$

74) a) We prove by induction.

Since no column sum is greater than r, no entry of $A^1 = A$ is greater than r.

Now suppose no entry of A^n is greater than r^n.

The value of $(A^{n+1})_{ij}$ is $\sum_{k=1}^{m} (A^n{}_{ik})(A_{kj})$. Since $A^n{}_{ik} \le r^n$ by our hypothesis, $(A^{n+1})_{ij} \le r^n \sum_{k=1}^{m} A_{kj} \le r^n(r)$ since no column sum of A is greater than r. Thus no entry of A^{n+1} is greater than r^{n+1}. By induction, the statement holds for all $n \ge 1$.

b) From above, $0 \le (A^n)_{ij} \le r^n < 1$ for all $n \ge 1$.

Thus $0 \le \lim_{n\to\infty} (A^n)_{ij} \le \lim_{n\to\infty} r^n = 0$ for $0 \le r < 1$ (by ratio test, or some other means)

By the sandwich theorem, $\lim_{n\to\infty} (A^n)_{ij} = 0$, so $\lim_{n\to\infty} A^n = 0$ (the zero matrix)

c) Let $B = I_m + \sum_{n=1}^{\infty} A^n$

Thus $B_{ij} = (I_m)_{ij} + \sum_{n=1}^{\infty} (A^n)_{ij}$

$\leq \sum_{n=0}^{\infty} r^n$ since each entry of A^n is less than r_i^n and all entries of I_m are less than or equal to 1.

But $|r|<1$, so this is a geometric series which converges.
Thus the sum $I_m + \sum_{n=1}^{\infty} A^n$ converges entry by entry.

d) $(I_m - A)(I_m + A + A^2 + \dots + A^n) = I_m - A + A - A^2 + A^2 - A^3 + \dots - A^{n-1} + A^{n-1} - A^n$
$= I_m - A^n$

$\lim_{n\to\infty} (I_m - A)(I_m + A + A^2 + \dots + A^n) = \lim_{n\to\infty} I_m - A^n = I_m$ (from (b) above).

Thus $(I_m - A)^{-1} = I_m + A + A^2 + \dots + A^n + \dots$ which makes sense, by (c) above.

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Homework #8 (TTh HW #6)

Solutions by Dave Freeman

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6) $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ $\quad x_2 = s,\ x_3 = t \longrightarrow \ker A = s\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

so $\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$ spans $\ker(A)$.

11) $\left[\begin{array}{cccc|c} 1 & 0 & 2 & 4 & 0 \\ 0 & 1 & -3 & -1 & 0 \\ 3 & 4 & -6 & 8 & 0 \\ 0 & -1 & 3 & 4 & 0 \end{array}\right] \begin{array}{l} \\ \\ -3R_1 \\ \\ \end{array} \longrightarrow \left[\begin{array}{cccc|c} 1 & 0 & 2 & 4 & 0 \\ 0 & 1 & -3 & -1 & 0 \\ 0 & 4 & -12 & -4 & 0 \\ 0 & -1 & 3 & 4 & 0 \end{array}\right] \begin{array}{l} \\ \\ -4R_2 \\ +R_2 \end{array} \longrightarrow \left[\begin{array}{cccc|c} 1 & 0 & 2 & 4 & 0 \\ 0 & 1 & -3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 \end{array}\right] \begin{array}{l} +\frac{4}{3}R_4 \\ +\frac{1}{3}R_4 \\ \\ \times -\frac{1}{3} \end{array}$

$\longrightarrow \left[\begin{array}{cccc|c} 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & -3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$ $\quad x_3 = t \longrightarrow \ker(A) = t\begin{pmatrix} -2 \\ 3 \\ 1 \\ 0 \end{pmatrix}$ so $\left\{ \begin{pmatrix} -2 \\ 3 \\ 1 \\ 0 \end{pmatrix} \right\}$ spans $\ker A$.

22) Are all the columns linearly independent? If they are not, then $\begin{pmatrix} 3 \\ 2 \\ 7 \end{pmatrix} = a_1 \begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix} + a_2 \begin{pmatrix} 1 \\ 4 \\ 5 \end{pmatrix}$ for some $a_1, a_2 \in \mathbb{R}$.

i.e. $\left[\begin{array}{cc|c} 2 & 1 & 3 \\ 3 & 4 & 2 \\ 6 & 5 & 7 \end{array}\right]$ has a solution.

$\left[\begin{array}{cc|c} 2 & 1 & 3 \\ 3 & 4 & 2 \\ 6 & 5 & 7 \end{array}\right] \begin{array}{l} \times\frac{1}{2} \\ -\frac{3}{2}R_1 \\ -3R_1 \end{array} \rightarrow \left[\begin{array}{cc|c} 1 & \frac{1}{2} & \frac{3}{2} \\ 0 & \frac{5}{2} & -\frac{5}{2} \\ 0 & 2 & -2 \end{array}\right] \begin{array}{l} -\frac{1}{4}R_3 \\ \times\frac{2}{5} \\ -\frac{4}{5}R_2 \end{array} \longrightarrow \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{array}\right]$

Thus the third column can be written as a linear combination of the first two. Since the first two columns are linearly independent (by inspection), the image is a two-dimensional subspace of $\mathbb{R}^3$ — i.e. a $\boxed{\text{plane in } \mathbb{R}^3}$

24) Image: the plane $x+2y+3z=0$

Kernel: all vectors normal to the plane $x+2y+3z=0$ (this set is $t\begin{pmatrix}1\\2\\3\end{pmatrix}$ for any t).

31) The plane with normal vector $\begin{pmatrix}1\\3\\2\end{pmatrix}$ has the equation $x+3y+2z=0$. Let y and z be free variables; then the plane is ~~spanned~~ given by $t\begin{pmatrix}-3\\1\\0\end{pmatrix}+s\begin{pmatrix}-2\\0\\1\end{pmatrix}$

Thus a matrix with the required image is

$$\begin{bmatrix}-3 & 2 & -1\\ 1 & 0 & 1\\ 0 & 1 & 1\end{bmatrix}.$$

(The third column can be any linear combination of the first two, including $\begin{pmatrix}0\\0\\0\end{pmatrix}$).

36) The cross product of $\vec{v}$ and $\vec{x}$ gives a vector perpendicular to both, whose magnitude is proportional to the sine of the angle between $\vec{v}$ and $\vec{x}$.

The image will be the set of all vectors perpendicular to $\vec{v}$ — a plane.

The kernel will be the set of all vectors parallel to $\vec{v}$ — the line spanned by $\vec{v}$.

39) a) Ker (B) is contained in ker(AB), because anything that B sends to zero, AB will send to zero as well; however, AB may send additional stuff to zero as well, so the kernels need not be equal.

A silly example: $A=\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}$ $B=\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}$.

b) Im (AB) is contained in Im(A), because anything that can come out of AB can also come out of A. However, the image of B need not be all of $\mathbb{R}^n$, so the ~~input~~ images need not be equal.

Example: $A=\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}$ $B=\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}$.

$\begin{pmatrix}0\\1\\0\end{pmatrix}$ is in the image of A, but not in the image of AB.

45) Ker (A) is the solution set to $A\vec{x} = \vec{0}$.

Thus Ker (A) is also the solution set to $(\text{rref } A)\vec{x} = \vec{0}$.

If we find rref A, we can parametrize the solutions to $(\text{rref } A)\vec{x} = 0$ in terms of the non-leading variables of rref A.

Since rank $A = r$, there are $n-r$ non-leading variables.

The solutions are thus $c_1\left(\vec{v}_1\right) + c_2\left(\vec{v}_2\right) + \cdots + c_{n-r}\left(\vec{v}_{n-r}\right)$

~~for~~ where $\vec{v}_i$ are fixed vectors and c_i are our parameters.

The kernel is thus ~~the~~ span of the $\vec{v}_i$, a set of $n-r$ vectors.

Math 21b
Homework #9 (TTh: HW #7)
Solutions by Dave Freeman

Sect. 3.2 #6, 18, 23, 24, 38, 40, 48, 50

6) a) $V \cap W$ is a subspace.
- $\vec{0} \in V$, $\vec{0} \in W \Rightarrow \vec{0} \in V \cap \vec{W}$.
- Let $\vec{x}, \vec{y} \in V \cap W$. Then $\vec{x}+\vec{y} \in V$ (since V a subspace), and $\vec{x}+\vec{y} \in W$ (since W a subspace) so $\vec{x}+\vec{y} \in V \cap W$.
- Let $\vec{x} \in V \cap W$, $k \in \mathbb{R}$. Then $k\vec{x} \in V$ (since V a subspace), $k\vec{x} \in W$ (since W a subspace), so $k\vec{x} \in V \cap W$.

b) $V \cup W$ is not necessarily a subspace.
Counterexample: Let $V = \text{span}\left(\begin{pmatrix}0\\1\end{pmatrix}\right) \subset \mathbb{R}^2$, and let $W = \text{span}\left(\begin{pmatrix}1\\0\end{pmatrix}\right) \subset \mathbb{R}^2$. Then $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$ are in $V \cup W$, but their sum $\begin{pmatrix}1\\1\end{pmatrix}$ is not.

18) $$x_1\begin{pmatrix}1\\1\\1\\1\end{pmatrix} + x_2\begin{pmatrix}1\\2\\3\\4\end{pmatrix} + x_3\begin{pmatrix}1\\4\\7\\10\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix} \Longrightarrow \begin{bmatrix}1&1&1\\1&2&4\\1&3&7\\1&4&10\end{bmatrix}\vec{x} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

$$\text{rref}\left[\begin{array}{ccc|c}1&1&1&0\\1&2&4&0\\1&3&7&0\\1&4&10&0\end{array}\right] \rightarrow \left[\begin{array}{ccc|c}1&1&1&0\\0&1&3&0\\0&2&6&0\\0&3&9&0\end{array}\right] \rightarrow \left[\begin{array}{ccc|c}1&0&-2&0\\0&1&3&0\\0&0&0&0\\0&0&0&0\end{array}\right]$$

so there are nontrivial solutions, so the vectors are <u>not</u> linearly independent.

23) $\vec{0} \in V^{\perp}$, since $\vec{0}\cdot\vec{v} = 0 \ \forall \ v \in V$.
Let $\vec{x}, \vec{y} \in V^{\perp}$. Then $(\vec{x}+\vec{y})\cdot\vec{v} = \vec{x}\cdot\vec{v} + \vec{y}\cdot\vec{v} = 0$ for all $\vec{v} \in V$.
Let $\vec{x}\cdot\vec{v}$, $k \in \mathbb{R}$. Then $(k\vec{x})\cdot\vec{v} = k(\vec{x}\cdot\vec{v}) = k\cdot 0 = 0$ for all $\vec{v} \in V$.
Thus $V^{\perp}$ is a subspace.
(Note: We're using the fact that the dot product is linear; this was shown in a previous HW.)

24) $\vec{x} \in L^{\perp} \Rightarrow \vec{x} \cdot \left(t\begin{pmatrix}1\\2\\3\end{pmatrix}\right) = 0 \Rightarrow \vec{x} \cdot \begin{pmatrix}1\\2\\3\end{pmatrix} = 0$

$\Rightarrow x_1 + 2x_2 + x_3 = 0.$ Let $x_2 = s$, $x_3 = t$; then

$\vec{x} = s\begin{pmatrix}-2\\1\\0\end{pmatrix} + t\begin{pmatrix}-1\\0\\1\end{pmatrix}$. $\left\{\begin{pmatrix}-2\\1\\0\end{pmatrix}, \begin{pmatrix}-1\\0\\1\end{pmatrix}\right\}$ spans $V^{\perp}$

and is clearly linearly independent, so it is a basis.

38) a) Suppose we could find more than n vectors that were linearly independent. Let $m>n$, and $\vec{v}_1, \dots, \vec{v}_m$ be the vectors.

Then $x_1\vec{v}_1 + x_2\vec{v}_2 + \dots + x_m\vec{v}_m = \vec{0}$ has a unique solution.

So the system $\begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_m \\ | & & | \end{bmatrix}\vec{x} = \vec{0}$ has a unique solution,

$\Rightarrow \operatorname{rref}\begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_m \\ | & & | \end{bmatrix}$ has a leading 1 in each column; i.e. $\operatorname{rank}\begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_m \\ | & & | \end{bmatrix} = m$.

But this is impossible, since $m>n$, and the rank can't be greater than the number of rows.

b) Proof by contradiction:

Suppose $\vec{v}_1, \dots, \vec{v}_m$ do not span V. Then there is some $\vec{w} \in V$ that is not a linear combination of the $\vec{v}_i$.

Thus $c_1\vec{v}_1 + \dots + c_m\vec{v}_m = \vec{w}$ has only the trivial solution $c_i = 0 \ \forall i$.

If $a_1\vec{v}_1 + \dots + a_m\vec{v}_m + a_{m+1}\vec{w} = \vec{0}$ has a non-trivial solution,

then $a_{m+1} \neq 0$, (since $\vec{v}_1, \dots, \vec{v}_m$ linearly independent),

and $\frac{-a_1}{a_{m+1}}\vec{v}_1 + \dots + \frac{-a_m}{a_{m+1}}\vec{v}_n = \vec{w}$ has a nontrivial solution,

which we just showed is impossible.

Thus $\vec{v}_1, \dots, \vec{v}_n, \vec{w}$ are linearly independent, contradicting our assumption that m is the largest possible number of linearly independent vectors in V.

We conclude that $\vec{v}_1, \dots, \vec{v}_m$ span V.

e) By part (b), every subspace has a basis.
Let $\vec{v}_1, \ldots, \vec{v}_n$ be a basis for V.
Let $A = \left[\vec{v}_1 \cdots \vec{v}_n \ \underbrace{\vec{0} \cdots \vec{0}}_{\text{some number of columns equal to } \vec{0}} \right]$. Then $\mathrm{Im}(A) = \mathrm{span}(\vec{v}_1, \ldots, \vec{v}_n) = V$.

40) Let $A = \left[\vec{a}_1 \cdots \vec{a}_n \right]$. If the $\vec{a}_i$ are linearly independent, then the system $A\vec{x} = \vec{0}$ has only the trivial solution, so $\ker A = \{\vec{0}\}$. Similarly, $\ker B = \{\vec{0}\}$.
Now suppose $AB\vec{x} = 0$; then $B\vec{x} \in \ker A \Rightarrow B\vec{x} = 0 \Rightarrow x \in \ker B \Rightarrow \vec{x} = 0$. Thus AB has kernel $\{0\}$, so its columns are linearly independent.

48) $\mathcal{E} = \ker [3 \ 4 \ 5]$
$\mathcal{E} = \mathrm{Span}\left\{ \begin{pmatrix} 4 \\ -3 \\ 0 \end{pmatrix}, \begin{pmatrix} 5 \\ 0 \\ -4 \end{pmatrix} \right\}$, so $\mathcal{E} = \mathrm{Im} \begin{bmatrix} 4 & 5 \\ -3 & 0 \\ 0 & -4 \end{bmatrix}$.

(since the solutions to $3x_1 + 4x_2 + 5x_3 = 0$ are $s \begin{pmatrix} -\frac{4}{3} \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -\frac{5}{3} \\ 0 \\ 1 \end{pmatrix}$.)

50) $\vec{0} \in V$, $\vec{0} \in W \Rightarrow \vec{0} + \vec{0} = \vec{0} \in V + W$.
suppose $\vec{v}_1, \vec{v}_2 \in V$, $\vec{w}_1, \vec{w}_2 \in W$. Then $(\vec{v}_1 + \vec{w}_1) + (\vec{v}_2 + \vec{w}_2)$
$= (\vec{v}_1 + \vec{v}_2) + (\vec{w}_1 + \vec{w}_2) \in V + W$ since $\vec{v}_1 + \vec{v}_2 \in V$ and $\vec{w}_1 + \vec{w}_2 \in W$.
suppose $\vec{v} \in V$, $\vec{w} \in W$, $k \in \mathbb{R}$. Then $k(\vec{v} + \vec{w}) = k\vec{v} + k\vec{w} \in V + W$ since $k\vec{v} \in V$ and $k\vec{w} \in W$.

Thus $V + W$ is a subspace.