HW #10

3.3/ 8, 20, 22, 24, 30, 31, 32, 44

8. $\begin{bmatrix} 1 & 2 & 3 & 2 & 1 \\ 3 & 6 & 9 & 6 & 3 \\ 1 & 2 & 4 & 1 & 2 \\ 2 & 4 & 9 & 1 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 3 & -3 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & 5 & -2 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

$\Rightarrow \ker(A) = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = t\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} -5 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + r\begin{bmatrix} 2 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$

$\left\{\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}\right\}$ is a basis of $\ker(A)$, so $\dim(\ker A) =$ nullity $(A) = 3$

20. $\operatorname{rref}(A) = \begin{bmatrix} 1 & 2 & 0 & 5 & -2 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ (see #8)

$\Rightarrow \left\{\begin{bmatrix} 1 \\ 3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 9 \\ 4 \\ 9 \end{bmatrix}\right\}$ is a ~~bases~~ basis for $\operatorname{Im}(A)$, so $\dim(\operatorname{Im} A) = \operatorname{rank}(A) = 2$

22. $\begin{bmatrix} 1 & 2 & 1 & 2 & 1 \\ 1 & 2 & 2 & 1 & 2 \\ 2 & 4 & 3 & 3 & 3 \\ 0 & 0 & 1 & -1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 & 2 & 1 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

A basis of $\operatorname{Im}(A)$ is $\left\{\begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ -1 \end{bmatrix}\right\} \Rightarrow \dim(\operatorname{Im} A) = 3$

A basis of $\ker(A)$ is $\left\{\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}\right\} \Rightarrow \dim(\ker A) = 2$

$\dim(\operatorname{Im} A) + \dim(\ker A) = 5$
= # of columns ~~so~~ so fact 3.3.9 holds

HW #10 (pg 2)

24. Form a 5x5 matrix with the vectors and find a basis for the image

$$\begin{bmatrix} 1 & 3 & 3 & 8 & 0 \\ 2 & 6 & 2 & 4 & 4 \\ 3 & 9 & 4 & 9 & 5 \\ 2 & 6 & 1 & 1 & 5 \\ 1 & 3 & 2 & 6 & 1 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 3 & 0 & -1 & 3 \\ 0 & 0 & 1 & 3 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 4 \\ 1 \\ 2 \end{bmatrix} \right\}$$ is a basis for the subspace spanned by the 5 vectors.

30. $2x_1 - x_2 + 2x_3 + 4x_4 = 0$

$x_1 = \frac{1}{2} x_2 - x_3 - 2x_4$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = r \begin{bmatrix} 1/2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$\Rightarrow \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for the subspace

31. First, find basis for the subspace V as in #30

$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for V

Let a be the 4x3 matrix with these column vectors

$A = \begin{bmatrix} 1 & -2 & -4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\quad$ $Im(A) = V$ + $Ker(A) = \{\vec{0}\}$ so A satisfies both requirements. note: this ~~the~~ solution is not unique.

32. Use the fact that perpendicular vectors have a dot product of zero. we have the following two constraints

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} = 0 \quad + \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix} = 0$$

which yields the system

$$\begin{Bmatrix} x_1 - x_3 + x_4 = 0 \\ x_2 + 2x_3 + 3x_4 = 0 \end{Bmatrix}$$

using an augmented matrix we solve the system

$$\left[\begin{array}{cccc|c} 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 2 & 3 & 0 \end{array}\right] \xrightarrow{\text{rref}} \left[\begin{array}{cccc|c} 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 2 & 3 & 0 \end{array}\right]$$

so a basis is $\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\}$

44. First show that $V \cap W = \{\vec{0}\}$ + $\dim V + \dim W = n$
$\Rightarrow V + W$ are complements

Choose a basis $\vec{v}_1 \dots \vec{v}_p$ of V
choose a basis $\vec{w}_1 \dots \vec{w}_q$ of W

Observe that $\vec{v}_1, \dots, \vec{v}_p, \vec{w}_1, \dots, \vec{w}_q$ are lin. indep.

Proof by contradiction:
Assume some $\vec{v}_i$ is lin dep. with $\vec{w}_1, \dots, \vec{w}_q$
$\Rightarrow \vec{v}_i = c_1 w_1 + \dots + c_q w_q$ for some $c_1, \dots, c_q \in \mathbb{R}$
$\Rightarrow \vec{v}_i \in W$
$\Rightarrow \vec{v}_i \in W \cap V$, an obvious contradiction

Further observe that $p + q = n$
Hence $\vec{v}_1, \dots, \vec{v}_p, \vec{w}_1, \dots, \vec{w}_q$ is a basis for $\mathbb{R}^n$
$\Rightarrow \vec{x} = c_1\vec{v}_1 + \dots + c_p\vec{v}_p + d_1\vec{w}_1 + \dots + d_q\vec{w}_q$ uniquely for all $\vec{x} \in \mathbb{R}^n$
Let $\vec{v} = c_1\vec{v}_1 + \dots + c_p\vec{v}_p$ and $\vec{w} = d_1\vec{w}_1 + \dots + d_q\vec{w}_q$
$\Rightarrow \vec{x} = \vec{v} + \vec{w}$ for a unique $\vec{w} \in W$ and $\vec{v} \in V$
$\Rightarrow V + W$ are complements.

Now, show that the converse is true.
we know that $W + V$ are complements
Observe that $W \cap V = \{\vec{0}\}$
Let $\vec{x} \in V \cap W$
we can write $\vec{x} = \underset{\text{in } V}{\vec{x}} + \underset{\text{in } W}{\vec{0}} = \underset{\text{in } V}{\vec{0}} + \underset{\text{in } W}{\vec{x}}$

Since this representation is unique (by def. of complements), $\vec{x}$ must equal $\vec{0}$ and we have that $V \cap W = \{\vec{0}\}$.

Finally observe that $\dim(V) + \dim(W) = n$

Let $\vec{v}_1, \dots, \vec{v}_p$ be a basis for V and $\vec{w}_1, \dots, \vec{w}_q$ be a basis for W

$\vec{v}_1, \dots, \vec{v}_p, \vec{w}_1, \dots, \vec{w}_q$ must span $\mathbb{R}^n$ since any $\vec{x} \in \mathbb{R}^n$ can be built with vectors in $V + W$

Further $\vec{v}_1, \dots, \vec{v}_p, \vec{w}_1, \dots, \vec{w}_q$ must be lin. indep. Otherwise some $\vec{v}_i$ would be in the intersection of $V + W$, a contradiction of $V \cap W = \{\vec{0}\}$ (this follows directly from reasoning done earlier in the proof)

$\vec{v}_1, \dots, \vec{v}_p, \vec{w}_1, \dots, \vec{w}_q$ span $\mathbb{R}^n$ and are lin indep.

$\Rightarrow \vec{v}_1, \dots, \vec{v}_p, \vec{w}_1, \dots, \vec{w}_q$ is a basis for $\mathbb{R}^n$

$\Rightarrow p + q = n$

$\Rightarrow \dim(V) + \dim(W) = n$

HW #11

4.1 / 6, 12, 16, 26

4.2 / 2, 4, 14

6. $\vec{u}\cdot\vec{v} = \|\vec{u}\|\|\vec{v}\|\cos\alpha$

$$\alpha = \cos^{-1}\left(\frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|}\right) = \cos^{-1}\left(\frac{2\cdot3+8\cdot10}{\sqrt{10}\,\sqrt{64}}\right) \approx 1.700 \text{ radians}$$

12. $$\|\vec{v}+\vec{w}\|^2 = (\vec{v}+\vec{w})\cdot(\vec{v}+\vec{w})$$
$$= \|\vec{v}\|^2+\|\vec{w}\|^2+2(\vec{v}\cdot\vec{w})$$
$$\leq \|\vec{v}\|^2+\|\vec{w}\|^2+2\|\vec{v}\|\|\vec{w}\| \text{ (by Cauchy-Schwarz)}$$
$$= (\|\vec{v}\|+\|\vec{w}\|)^2$$

So, $\|\vec{v}+\vec{w}\|^2 \leq (\|\vec{v}\|+\|\vec{w}\|)^2$

taking the square root of both sides, we have

$\|\vec{v}+\vec{w}\| \leq \|\vec{v}\|+\|\vec{w}\|$

16. Any vector perpendicular to $\vec{v}_1, \vec{v}_2, \vec{v}_3$ will be of the form $\begin{bmatrix} a \\ -a \\ -a \\ a \end{bmatrix}$

* To be a unit vector a must equal $\pm 1/2$

So we have two possible vectors: $\begin{bmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{bmatrix}$ and $\begin{bmatrix} -1/2 \\ 1/2 \\ 1/2 \\ -1/2 \end{bmatrix}$

26. The two vectors are orthognal, but not unit vectors. To get an orthonormal basis, divide by 7

$$\vec{v}_1 = \frac{1}{7}\begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix}, \quad \vec{v}_2 = \frac{1}{7}\begin{bmatrix} 3 \\ -6 \\ 2 \end{bmatrix}$$

$$\text{proj}_V \begin{bmatrix} 49 \\ 49 \\ 49 \end{bmatrix} = \left(\vec{v}_1\cdot\begin{bmatrix} 49 \\ 49 \\ 49 \end{bmatrix}\right)\vec{v}_1 + \left(\vec{v}_2\cdot\begin{bmatrix} 49 \\ 49 \\ 49 \end{bmatrix}\right)\vec{v}_2 = \begin{bmatrix} 19 \\ 39 \\ 64 \end{bmatrix}$$

HW #11 (Pg. 2)

2. $\vec{w}_1 = \dfrac{\vec{v}_1}{\|\vec{v}_1\|} = \dfrac{1}{7}\begin{bmatrix}6\\3\\2\end{bmatrix}$

$\vec{w}_2 = \dfrac{\vec{v}_2 - (\vec{w}_1 \cdot \vec{v}_2)\,\vec{w}_1}{\text{length}} = \dfrac{1}{7}\begin{bmatrix}2\\-6\\3\end{bmatrix}$

4. $\vec{w}_1 = \dfrac{\vec{v}_1}{\|\vec{v}_1\|} = \dfrac{1}{5}\begin{bmatrix}4\\0\\3\end{bmatrix}$

$\vec{w}_2 = \dfrac{\vec{v}_2 - (\vec{w}_1 \cdot \vec{v}_2)\,\vec{w}_1}{\text{length}} = \dfrac{1}{5}\begin{bmatrix}3\\0\\-4\end{bmatrix}$

Since $\vec{v}_3$ is orthognal to $\vec{w}_1$ + $\vec{w}_2$,

$\vec{w}_3 = \dfrac{\vec{v}_3}{\|\vec{v}_3\|} = \begin{bmatrix}0\\-1\\0\end{bmatrix}$

14. $\vec{w}_1 = \dfrac{\vec{v}_1}{\|\vec{v}_1\|} = \dfrac{1}{10}\begin{bmatrix}1\\7\\1\\7\end{bmatrix}$

$\vec{w}_2 = \dfrac{\vec{v}_2 - (\vec{w}_1 \cdot \vec{v}_2)\,\vec{w}_1}{\text{length}} = \dfrac{1}{\sqrt{2}}\begin{bmatrix}-1\\0\\1\\0\end{bmatrix}$

$\vec{w}_3 = \dfrac{\vec{v}_3 - (\vec{w}_1 \cdot \vec{v}_3)\,\vec{w}_1 - (\vec{w}_2 \cdot \vec{v}_3)\,\vec{w}_2}{\text{length}} = \dfrac{1}{\sqrt{2}}\begin{bmatrix}0\\1\\0\\-1\end{bmatrix}$

HW #12

4.2/ 18, 28, 34

4.3/ 8, 9, 11, 18, 20

18. $$Q = \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ \vec{w}_1 & \vec{w}_2 & \vec{w}_3 \\ \downarrow & \downarrow & \downarrow \end{bmatrix}$$

refer to 4.2.4 (HW #10) for $\vec{w}_1, \vec{w}_2, \vec{w}_3$

$$Q = \frac{1}{5}\begin{bmatrix} 4 & 3 & 0 \\ 0 & 0 & -5 \\ 3 & -4 & 0 \end{bmatrix}$$

$$R = \begin{bmatrix} \|\vec{v}_1\| & \vec{w}_1 \cdot \vec{v}_2 & \vec{w}_1 \cdot \vec{v}_3 \\ 0 & \|\vec{v}_2 - (\vec{w}_1 \cdot \vec{v}_2)\vec{w}_1\| & \vec{w}_2 \cdot \vec{v}_3 \\ 0 & 0 & \|\vec{v}_3\| \end{bmatrix} = \begin{bmatrix} 5 & 5 & 0 \\ 0 & 35 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

28. from 4.2.14 (HW #10)

$$Q = \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ \vec{w}_1 & \vec{w}_2 & \vec{w}_3 \\ \downarrow & \downarrow & \downarrow \end{bmatrix} = \begin{bmatrix} 1/10 & -1/\sqrt{2} & 0 \\ 7/10 & 0 & 1/\sqrt{2} \\ 1/10 & 1/\sqrt{2} & 0 \\ 7/10 & 0 & -1/\sqrt{2} \end{bmatrix}$$

$$R = \begin{bmatrix} 10 & 10 & 10 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix}$$

34. $$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \end{bmatrix}$$

$$\Rightarrow \left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\}$$ is a basis for $\ker(A)$

8. a. No. Consider the counter example

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$A A^T$ is not defined

b. Yes. If A + B are $n \times n$ matrices and ~~ABn~~ $BA = I_n$ then $B = A^{-1}$.

Hence $A^T = A^{-1}$ and $AA^T = AA^{-1} = I_n$

9. Write $A = [\vec{v}_1 \ \vec{v}_2]$. we can express the unit vector $\vec{v}_1$ as $\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$ for some θ. $\vec{v}_2$ must be a unit vector perpendicular to $\vec{v}_1$, so $\vec{v}_2 = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$ or $\vec{v}_2 = \begin{bmatrix} \sin\theta \\ -\cos\theta \end{bmatrix}$

$\therefore$ an orthognal matrix is of the form $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ or $A = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$, representing a rotation or reflection.

11. First, think of the inverse $L = T^{-1}$ of T

$L(x) = A\vec{x} = [\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3] \ \vec{x}$.

we know that $L(\vec{e}_3) = \vec{v}_3 = \begin{bmatrix} 2/3 \\ 2/3 \\ 1/3 \end{bmatrix}$

$\vec{v}_1, \vec{v}_2, \vec{v}_3$ must be an orthonormal basis of $\mathbb{R}^3$

Arbitrarily, we choose $\vec{v}_1 = \begin{bmatrix} -2/3 \\ 1/3 \\ 2/3 \end{bmatrix}$ (this is orthonormal by inspection

Now choose $\vec{v}_2 = \vec{v}_1 \times \vec{v}_3$.

we have $A = \frac{1}{3}\begin{bmatrix} -2 & -1 & 2 \\ 1 & 2 & 2 \\ 2 & -2 & 1 \end{bmatrix}$

Since ~~this~~ A is orthognal the matrix of $T = L^{-1}$ is A^T

$T(\vec{x}) = \frac{1}{3}\begin{bmatrix} -2 & 1 & 2 \\ -1 & 2 & -2 \\ 2 & 2 & 1 \end{bmatrix} \vec{X}$ note: this solution is not unique.

18.a. $a_{ij} = -a_{ji}$

note: diagonal entries must be zero since $a_{ii} = -a_{ii}$

One 3×3 example:

$$\begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & -3 \\ -2 & 3 & 0 \end{bmatrix}$$

b. $(A^2)^T = (AA)^T = A^T A^T = (-A)(-A) = A^2$

so A^2 is symmetric

20. First, use Gram-Schmidt to find the orthonormal basis

$$\vec{w}_1 = \begin{bmatrix} .5 \\ .5 \\ .5 \\ .5 \end{bmatrix} \quad \vec{w}_2 = \begin{bmatrix} -.1 \\ .7 \\ -.7 \\ .1 \end{bmatrix}$$

If $A = [\vec{w}_1 \ \vec{w}_2]$, ~~projection onto W is given~~

the matrix of projection onto W is

$$AA^T = \frac{1}{100}\begin{bmatrix} 26 & 18 & 32 & 24 \\ 18 & 74 & -24 & 32 \\ 32 & -24 & 74 & 18 \\ 24 & 32 & 18 & 26 \end{bmatrix}$$