Math 21b Solutions; HW #13

4.3

34) Given $M, N \in H$ w/ $M = \begin{pmatrix} p & -q & -r & -s \\ q & p & s & -r \\ r & -s & p & q \\ s & r & -q & p \end{pmatrix}$, $N = \begin{pmatrix} a & -b & -c & -d \\ b & a & d & -c \\ c & -d & a & b \\ d & c & -b & a \end{pmatrix}$

a) Let $p' = p + a$, $q' = q + b$, $r' = r + c$, $s' = s + d$. Then,

$$M + N = \begin{pmatrix} p' & -q' & -r' & -s' \\ q' & p' & s' & -r' \\ r' & -s' & p' & q' \\ s' & r' & -q' & p' \end{pmatrix} \in \mathcal{H}$$

b) Given $k \in \mathbb{R}$, Let $p' = kp$, $q' = kp$, $r' = kr$, $s' = kr$. Then,

$$kM = \begin{pmatrix} p' & -q' & -r' & -s' \\ q' & p' & s' & -r' \\ r' & -s' & p' & q' \\ s' & r' & -q' & p' \end{pmatrix} \in \mathcal{H}$$

c) Let
$p' = pa - qb - rc - sd$
$q' = qa + pb + sc - rd$
$r' = ra - sb + pc + qd$
$s' = sa + rb - qc + pd$
Then,

$$MN = \begin{pmatrix} p' & -q' & -r' & -s' \\ q' & p' & s' & -r' \\ r' & -s' & p' & q' \\ s' & r' & -q' & p' \end{pmatrix} \in \mathcal{H}$$

d) $$\begin{pmatrix} A & -B^T \\ B & A^T \end{pmatrix}^T = \begin{pmatrix} A^T & B^T \\ (-B^T)^T & (A^T)^T \end{pmatrix} = \begin{pmatrix} A^T & B^T \\ -B & A \end{pmatrix} \in \mathcal{H}$$

e) $$M^T M = (p^2 + q^2 + r^2 + s^2) I_4$$

f) if $M \neq 0$, $\left(\frac{1}{p^2+q^2+r^2+s^2} M^T\right) M = I_4$

$\therefore$ as M is square & has a left inverse, it is invertible w/ $M^{-1} = \frac{1}{p^2+q^2+r^2+s^2} M^T$

by (b) & (d), $M^{-1} \in \mathcal{H}$.

g) No. Consider $M = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$ & $N = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$. $MN \neq NM$.

4.4

10)

a) Given $\vec{x}$ s.t. $A\vec{x} = \vec{b}$ (such $\vec{x}$ exists as $A\vec{x} = \vec{b}$ is consistent)

Let $\vec{x}_h = \operatorname{proj}_{\ker(A)} \vec{x}$

Let $\vec{x}_0 = \vec{x} - \vec{x}_h$.

Claim: $A\vec{x}_0 = \vec{b}$

Pf: $A\vec{x}_0 = A(\vec{x} - \vec{x}_h)$
$= A\vec{x} - A\vec{x}_h$
$= \vec{b} - \vec{0}$ as $\vec{x}_h \in \ker(A)$
$= \vec{b}$.

Claim: $\vec{x}_0 \in (\ker(A))^\perp$

Pf:
$\vec{x}_h = \operatorname{proj} \vec{x}$
$\vec{x}_0 = \vec{x} - \operatorname{proj} \vec{x} \Rightarrow \vec{x}_0 \perp \vec{x}_h$

$\therefore$ as $\vec{x}_h \in \ker(A)$, $\vec{x}_0 \in (\ker(A))^\perp$ ▨

b) Suppose $\vec{x}, \vec{y} \in (\ker(A))^\perp$ & $A\vec{x} = A\vec{y} = b$.

Now, $\ker(A)$ is a subspace by Fact 3.2.2

$\Rightarrow (\ker(A))^\perp$ is a subspace by Fact 4.1.5

$\therefore$ $\vec{x}, \vec{y} \in (\ker(A))^\perp \Rightarrow \vec{x} - \vec{y} \in (\ker(A))^\perp$

Furthermore, $A(\vec{x} - \vec{y}) = A\vec{x} - A\vec{y} = 0$

$\therefore$ $\vec{x} - \vec{y} \in \ker(A)$.

Hence, $\vec{x} - \vec{y} = 0$. $\therefore$ $\vec{x} = \vec{y}$ ▨

c) Given $\vec{x}_1$ s.t. $A\vec{x}_1 = \vec{b}$.

Let $\vec{x}_0$ be the solution to $A\vec{x} = \vec{b}$ in $(\ker(A))^\perp$.

by part (a) $\vec{x}_1 = \vec{x}_0 + \vec{x}_h$ w/ $\vec{x}_0 \perp \vec{x}_h$

$\therefore$ by Pythagorean Thm $\|\vec{x}_1\|^2 = \|\vec{x}_0\|^2 + \|\vec{x}_h\|^2$

$\Rightarrow \|\vec{x}_1\|^2 \geq \|\vec{x}_0\|^2$

$\Rightarrow \|\vec{x}_1\| \geq \|\vec{x}_0\|$

▨

4.4

22) $\vec{X}^* = (A^TA)^{-1}A^T\vec{b} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}.$

$A\vec{x}^* = b \Rightarrow \|\vec{b} - A\vec{x}^*\| = 0.$

26) $A^TA\vec{x}^* = A^T\vec{b}$

$$\Rightarrow \begin{pmatrix} 66 & 78 & 90 \\ 78 & 93 & 108 \\ 90 & 108 & 126 \end{pmatrix}\begin{pmatrix} x_1^* \\ x_2^* \\ x_3^* \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \Rightarrow \vec{X}^* = \begin{pmatrix} t - 7/6 \\ 1-2t \\ t \end{pmatrix}$$

where t is an arbitrary constant.

32)

$$\begin{array}{ccc} 1 & t & t^2 \end{array}$$
$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix}\begin{pmatrix} c_0 \\ c_1 \\ c_2 \end{pmatrix} = \begin{array}{c} f(t) \\ \begin{pmatrix} 27 \\ 0 \\ 0 \\ 0 \end{pmatrix} \end{array}$$

$$\therefore \begin{pmatrix} c_0^* \\ c_1^* \\ c_2^* \end{pmatrix} = \left[\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix}^T \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix}\right]^{-1} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix}^T \begin{pmatrix} 27 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 25.65 \\ -28.35 \\ 6.75 \end{pmatrix}$$

$\therefore f(t) \approx 25.65 - 28.35t + 6.75t^2.$

41) a) We want $\begin{pmatrix} c_0 \\ c_1 \end{pmatrix}$ s.t. $\log D = c_0 + c_1 t$ (t in years since 1970)

$$\begin{pmatrix} 1 & 0 \\ 1 & 5 \\ 1 & 10 \\ 1 & 15 \\ 1 & 20 \\ 1 & 25 \end{pmatrix}\begin{pmatrix} c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} \log 370 \\ \log 533 \\ \log 908 \\ \log 1823 \\ \log 3233 \\ \log 4871 \end{pmatrix} \Rightarrow \begin{pmatrix} c_0 \\ c_1 \end{pmatrix}^* \approx \begin{pmatrix} 2.53 \\ 0.047 \end{pmatrix}$$

$\therefore \log D \approx 2.53 + 0.047t$

$\Rightarrow D \approx 339 \cdot 10^{0.047t}$

b) in 2020, $t = 50 \Rightarrow D \approx 76{,}000.$

Projected debt is \$76 trillion.

5.1

16) $$\det\begin{pmatrix} 0&0&2&3&1\\ 0&0&0&2&2\\ 0&9&7&9&3\\ 0&0&0&0&5\\ 3&4&5&8&5 \end{pmatrix} = 3\det\begin{pmatrix} 0&2&3&1\\ 0&0&2&2\\ 9&7&9&3\\ 0&0&0&5 \end{pmatrix} = 3\cdot 5\cdot \det\begin{pmatrix} 0&2&3\\ 0&0&2\\ 9&7&9 \end{pmatrix}$$

$$= -3\cdot 5\cdot 2\cdot \det\begin{pmatrix} 0&2\\ 9&7 \end{pmatrix} = -30[0\cdot 7 - 9\cdot 2] = \boxed{540}$$

21) $\det\begin{pmatrix} 7&6\\ 9&8 \end{pmatrix} = 7\cdot 8 - 6\cdot 9 = 2.$ $\det \neq 0$ $\therefore$ the matrix is invertible

22) $\det\begin{pmatrix} 3&2\\ 6&4 \end{pmatrix} = 12-12=0.$ $\therefore$ the matrix is <u>not</u> invertible.

23) $\det\begin{pmatrix} a&b&c\\ 0&b&c\\ 0&0&c \end{pmatrix} = abc$ $\therefore$ invertible iff $a\neq 0, b\neq 0, \& \ c\neq 0$

24) $\det\begin{pmatrix} 1&2&3\\ 4&5&6\\ 7&8&9 \end{pmatrix} = 0$ $\therefore$ not invertible

25) $\det\begin{pmatrix} 1&2&3\\ 4&5&6\\ 7&8&7 \end{pmatrix} = 6$ $\therefore$ invertible

28) $\det(\lambda I - A) = \det\begin{pmatrix} \lambda-2 & 3\\ -3 & \lambda-2 \end{pmatrix} = \lambda^2+4\lambda+13 > 0 \quad \forall \lambda\in\mathbb{R}$

$\therefore$ $\lambda I - A$ is invertible for all real λ.

46) $\det[\vec{u}\ \vec{v}\ \vec{w}] = \vec{u}\cdot(\vec{v}\times\vec{w})$ for any 3x3 matrix $[\vec{u}\ \vec{v}\ \vec{w}]$.

$\therefore$ $\det(A) = (\vec{v}\times\vec{w})\cdot(\vec{v}\times\vec{w}) = \|\vec{v}\times\vec{w}\|^2$.

A is invertible if $\vec{v}\times\vec{w}\neq 0$, i.e. if $\vec{v}$ & $\vec{w}$ are not parallel.

5.2

6) $$\det(M_n) = \det\begin{pmatrix} 1&1&\cdots&1\\ 1&2&\cdots&2\\ \vdots&&&\vdots\\ 1&2&3\,\cdots&n \end{pmatrix} = \det\begin{pmatrix} 1&1&\cdots&1\\ 1&2&\cdots&2\\ \vdots&&&\vdots\\ 1&2&&n-1\\ 0&0&\cdots 0&1 \end{pmatrix}$$ we obtain this matrix by subtracting next to last row from the last row

$$= \det(M_{n-1})$$

But, $\det(M_1) = \det(1) = 1$

$\therefore$ $\boxed{\det(M_n) = 1 \quad \forall n\in\mathbb{N}}$

5.2

20) a) A is invertible

$\Rightarrow \text{rref}(A) = I_n$

$\Rightarrow \text{rref}[A \vdots AB] = [I_n \vdots M]$ for some matrix M.

Now, $\forall i \quad [A \vdots AB]\begin{bmatrix} B\vec{e_i} \\ -\vec{e_i} \end{bmatrix} = ABe_i - AB\vec{e_i} = 0$

$\therefore \begin{bmatrix} B\vec{e_i} \\ -\vec{e_i} \end{bmatrix} \in \ker[A \vdots AB] \quad \forall i$

$\Rightarrow \begin{bmatrix} B\vec{e_i} \\ -\vec{e_i} \end{bmatrix} \in \ker[I_n \vdots M] \quad \forall i$

$\therefore [I_n \vdots M]\begin{bmatrix} B\vec{e_i} \\ -\vec{e_i} \end{bmatrix} = B\vec{e_i} - M\vec{e_i} = 0 \quad \forall i$

$\Rightarrow B\vec{e_i} = M\vec{e_i} \quad \forall i$

Thus, B & M have identical columns.

Hence, $B = M$.

$\therefore \text{rref}[A \vdots AB] = [I_n \vdots B]$ ∎

b) if $B = A^{-1}$ we get

$\text{rref}[A \vdots I_n] = [I_n \vdots A^{-1}]$

which justifies the method of finding an inverse that we have been using.

5.2

8) $T(\vec{x}) = 0$ iff $\vec{x}$ is a lin. combination of V_i's
i.e iff $\vec{x} \in \text{span}\{\vec{v_2}, \dots, \vec{v_n}\}$.

$\therefore$ $\boxed{\ker(T) = \text{span}\{\vec{v_2}, \dots, \vec{v_n}\}.}$

det is a mapping from the space of $n \times n$ matrices to $\mathbb{R}$

$\therefore$ $\boxed{\text{Im}(T) = \mathbb{R}}$

Hence,

$$\boxed{\begin{array}{l} \dim(\ker(T)) = n-1 \\ \dim(\text{Im}(T)) = 1 \end{array}}$$

30) a) Claim: $\vec{v_2} \times \vec{v_3} \times \dots \times \vec{v_n} \neq 0$ iff $\{\vec{v_2}, \vec{v_3}, \dots, \vec{v_n}\}$ is lin. ind.

Suppose $\{\vec{v_2}, \vec{v_3}, \dots, \vec{v_n}\}$ are lin. ind. Choose an $\vec{x} \notin \text{span}\{\vec{v_2}, \dots, \vec{v_n}\}$.
So, $\{\vec{x}, \vec{v_2}, \dots, \vec{v_n}\}$ is a basis of $\mathbb{R}^n$

$\therefore$ $\det[\vec{x}\ \vec{v_2} \cdots \vec{v_n}] = \vec{x} \cdot (\vec{v_2} \times \vec{v_3} \times \dots \times \vec{v_n}) \neq 0$

$\Rightarrow$ $\vec{v_2} \times \vec{v_3} \times \dots \times \vec{v_n} \neq 0$

Now, suppose $\vec{v_2} \times \vec{v_3} \times \dots \times \vec{v_n} \neq 0$. Say the ith component of this vector is non-zero. Then $\vec{e_i} \cdot (\vec{v_2} \times \dots \times \vec{v_n}) = \det[\vec{e_i}\ \vec{v_2} \cdots \vec{v_n}] \neq 0$

$\therefore$ $\{\vec{v_2}, \dots \vec{v_n}\}$ is lin. ind.

b) $\vec{e_2} \times \vec{e_3} \times \dots \times \vec{e_n} = \det\left[\vec{e_i}\ \vec{e_2} \cdots \vec{e_n}\right] = \begin{cases} 1 & \text{if } i=1 \\ 0 & \text{if } i>1 \end{cases}$

$\therefore$ $\vec{e_2} \times \vec{e_3} \times \dots \times \vec{e_n} = \vec{e_1}$

c) $\vec{v_i} \cdot (\vec{v_2} \times \dots \times \vec{v_n}) = \det(\vec{v_i}\ \vec{v_2} \cdots \vec{v_n}) = 0$ $\quad \forall i \in \{2, 3, \dots, n\}$ as the matrix has 2 identical columns.

d) $\det\left(\vec{e_i}\ \vec{v_2}\ \vec{v_3} \cdots \vec{v_n}\right) = -\det\left(\vec{e_i}\ \vec{v_3}\ \vec{v_2} \cdots \vec{v_n}\right)$

$\therefore$ $\vec{v_2} \times \vec{v_3} \times \dots \times \vec{v_n} = -\vec{v_3} \times \vec{v_2} \times \dots \times \vec{v_n}$

HW #15

5.2

30)

e) $\det\begin{bmatrix} \vec{v}_2\times\vec{v}_3\times\cdots\times\vec{v}_n & \vec{v}_2 & \vec{v}_3 & \cdots & \vec{v}_n\end{bmatrix} = (\vec{v}_2\times\vec{v}_3\times\cdots\times\vec{v}_n)\cdot(\vec{v}_2\times\vec{v}_3\times\cdots\times\vec{v}_n)$

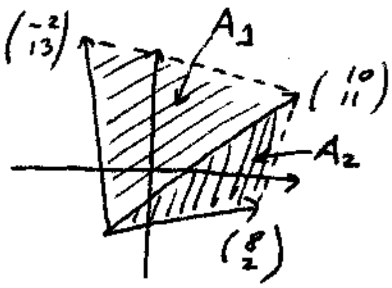
$= \|\vec{v}_2\times\vec{v}_3\times\cdots\times\vec{v}_n\|^2.$

f) see page 240 of Bretscher.

5.3

2) By Fact 5.3.3 $\text{Area} = \frac{1}{2}\left|\det\begin{pmatrix}3 & 8\\ 7 & 2\end{pmatrix}\right| = \frac{1}{2}|-50| = \boxed{25}$

7)



$\text{Area} = A_1 + A_2$

$= \frac{1}{2}\left|\det\begin{pmatrix}10 & -2\\ 11 & 13\end{pmatrix}\right| + \frac{1}{2}\left|\det\begin{pmatrix}8 & 10\\ 2 & 11\end{pmatrix}\right| = \boxed{110}.$

14) By Fact 5.3.7 $V = \left\{\det\left[\begin{pmatrix}1&0&0&0\\1&1&1&1\\1&2&3&4\end{pmatrix}\begin{pmatrix}1&1&1\\0&1&2\\0&1&3\\0&1&4\end{pmatrix}\right]\right\}^{1/2} = \left\{\det\begin{pmatrix}1&1&1\\1&4&10\\1&10&30\end{pmatrix}\right\}^{1/2}$

$= \boxed{\sqrt{6}}$

17) a) Let $\vec{w} = \vec{v}_1\times\vec{v}_2\times\vec{v}_3$.

Note that $\vec{w}\perp\vec{v}_i$ for $i=1,2,3$. (see 5.20.30c)

Now, $V(\vec{v}_1,\vec{v}_2,\vec{v}_3,\vec{w}) = V(\vec{v}_1,\vec{v}_2,\vec{v}_3)\,\|\vec{w} - \text{proj}_{V_3}\vec{w}\|$ by Def 5.3.6

$= V(\vec{v}_1,\vec{v}_2,\vec{v}_3)\,\|\vec{w}\|$ as $\text{proj}_{V_3}\vec{w} = 0$ since $\vec{w}\perp v_i\ \forall i$.

b) $V(v_1, v_2, v_3, \vec{v}_1\times\vec{v}_2\times\vec{v}_3) = \left|\det\begin{bmatrix}\vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{v}_1\times\vec{v}_2\times\vec{v}_3\end{bmatrix}\right|$

$= \left|\det\begin{bmatrix}\vec{v}_1\times\vec{v}_2\times\vec{v}_3 & \vec{v}_1 & \vec{v}_2 & \vec{v}_3\end{bmatrix}\right|$

$= \|v_1\times v_2\times v_3\|^2$ by 5.2.30e

c) by a & b $V(\vec{v}_1,\vec{v}_2,\vec{v}_3) = \|\vec{v}_1\times\vec{v}_2\times\vec{v}_3\|$.

if $\{\vec{v}_1,\vec{v}_2,\vec{v}_3\}$ is lin. dependent, both sides of the equation are 0 by Exercise 15 & Exercise 5.2.30a

HW #15

<u>5.3</u>

25) $[\text{adj}(A)]_{ij} = (-1)^{i+j}\det(A_{ji})$

$$\Rightarrow \text{adj}(A) = \begin{pmatrix} 1 & 0 & -1 \\ 0 & -1 & 0 \\ -2 & 0 & 1 \end{pmatrix}.$$

$$\therefore \quad A^{-1} = \frac{1}{\det(A)}\,\text{adj}(A) = \frac{1}{-1}\begin{pmatrix} 1 & 0 & -1 \\ 0 & -1 & 0 \\ -2 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & -1 \end{pmatrix}.$$