Math 21 b
Homework #16 (TTh: HW #12)
Solutions by Dave Freeman

Sect. 6.1

10) $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \end{bmatrix} \Rightarrow \begin{matrix} a+2b=5 \\ c+2d=10 \end{matrix} \Rightarrow \begin{matrix} a = 5-2b \\ c = 10-2d \end{matrix}$

so all such matrices are $\begin{bmatrix} 5-2b & b \\ 10-2d & d \end{bmatrix}$ for $b, d \in \mathbb{R}$.

23) a) $S\vec{e}_i = \vec{v}_i$, so $S^{-1}\vec{v}_i = S^{-1}(S\vec{e}_i) = \vec{e}_i$.

b) $\vec{v}_i = S\vec{e}_i = i^{th}$ column of S.

$S^{-1}AS\vec{e}_i = S^{-1}A\vec{v}_i = S^{-1}(\lambda_i\vec{v}_i) = \lambda_i\vec{e}_i$ by part (a).

Thus the i^{th} column of $S^{-1}AS$ is $\lambda_i\vec{e}_i$, so

$$S^{-1}AS = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

34) $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 15 \\ 5 \end{bmatrix} \Rightarrow \begin{matrix} 3a+b = 15 \\ 3c+d = 5 \end{matrix}$

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 10 \\ 20 \end{bmatrix} \Rightarrow \begin{matrix} a+2b = 10 \\ c+2d = 20 \end{matrix}$

$$\left[\begin{array}{cccc|c} 3 & 1 & 0 & 0 & 15 \\ 1 & 2 & 0 & 0 & 10 \\ 0 & 0 & 3 & 1 & 5 \\ 0 & 0 & 1 & 2 & 20 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & 11 \end{array}\right]$$ so the matrix is $\begin{bmatrix} 4 & 3 \\ -2 & 11 \end{bmatrix}$

37) a) $\begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{bmatrix}$ is a 2×2 orthogonal matrix representing reflection about some line. It thus has eigenvalues ± 1.

$\begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}$ is a reflection followed by a dilation by a factor of 5, so its eigenvalues are ± 5

b) Find $\ker(\lambda I_2 - A)$:

$\lambda = 5$ $\left[\begin{array}{cc|c} 2 & -4 & 0 \\ -4 & 8 & 0 \end{array}\right] \rightarrow$ eigenvector $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\lambda = -5$ $\left[\begin{array}{cc|c} -8 & -4 & 0 \\ -4 & -2 & 0 \end{array}\right] \rightarrow$ eigenvector $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$

39) Suppose λ is an eigenvalue of A.
Then $A\vec{v} = \lambda\vec{v}$ for some $\vec{v}$ (nonzero).
$(S^{-1}AS)(S^{-1}\vec{v}) = S^{-1}A\vec{v} = S^{-1}\lambda\vec{v} = \lambda(S^{-1}\vec{v})$
so $S^{-1}\vec{v}$ is an eigenvector of $S^{-1}AS$ with eigenvalue λ.
Now suppose μ is an eigenvalue of $S^{-1}AS$.
Then $S^{-1}AS\vec{w} = \mu\vec{w}$ for some nonzero $\vec{w}$.
$S(S^{-1}AS)\vec{w} = S(\mu\vec{w}) \Rightarrow A(S\vec{w}) = \mu(S\vec{w})$
So $S\vec{w}$ is an eigenvector of A with eigenvalue μ.
Thus A and $S^{-1}AS$ have the same eigenvalues.

Sect. 6.2

10) $\det(\lambda I_3 - A) = 0 \Rightarrow \det\begin{pmatrix} \lambda+3 & 0 & -4 \\ 0 & \lambda+1 & 0 \\ 2 & -7 & \lambda-3 \end{pmatrix} = 0$

expand across middle row:
$(\lambda+1)\big((\lambda+3)(\lambda-3)+8\big) = 0 \Rightarrow (\lambda+1)(\lambda^2-1) = 0 \Rightarrow (\lambda+1)^2(\lambda-1) = 0$
so the eigenvalues are -1 (multiplicity 2) and 1 (multiplicity 1)

24) $\det(\lambda I_2 - A) = 0 \Rightarrow \det\begin{pmatrix} \lambda-.5 & -.25 \\ -.5 & \lambda-.75 \end{pmatrix} = 0 \Rightarrow (\lambda-.5)(\lambda-.75) - .125 = 0$

$\Rightarrow \lambda^2 - 1.25\lambda + .25 = 0 \Rightarrow (\lambda-1)(\lambda-.25) = 0$ so $\lambda = 1$ or $.25$.

25) $\begin{bmatrix} a & c \\ b & d \end{bmatrix}\begin{bmatrix} c \\ b \end{bmatrix} = \begin{bmatrix} ac+bc \\ bc+db \end{bmatrix} = \begin{bmatrix} c \\ b \end{bmatrix}$ since $a+b = c+d = 1$

so $\begin{bmatrix} c \\ b \end{bmatrix}$ is an eigenvector with eigenvalue 1.

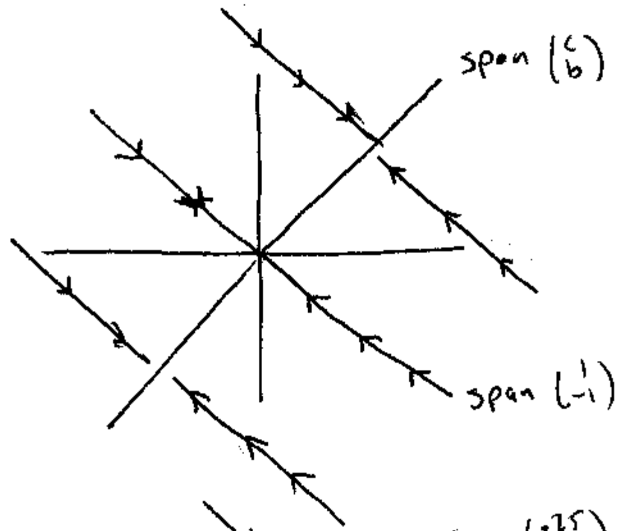
$\begin{bmatrix} a & c \\ b & d \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} a-c \\ b-d \end{bmatrix} = (a-c)\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ since $a+b = c+d$
$\Rightarrow b-d = -(a-c)$

so $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is an eigenvector with eigenvalue $a-c$

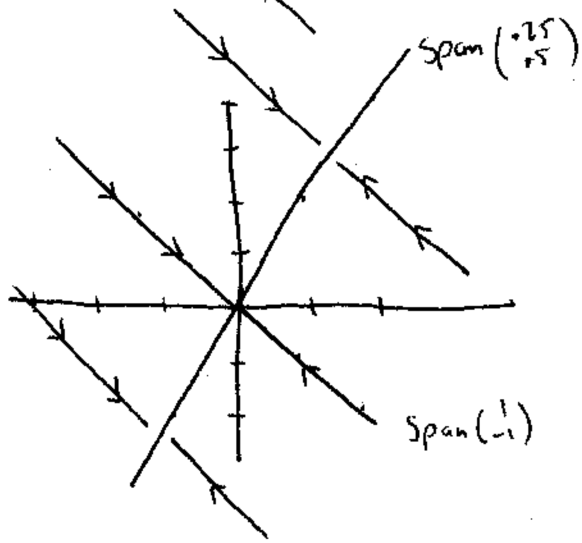
$|a-c| < 1$ since a, c both positive and ≤ 1.

25) (cont'd)

phase portrait:



26) $\binom{c}{b} = \binom{.25}{.5}$



27) a) $\vec{e}_1 = \binom{1}{0} = c_1 \binom{.25}{.5} + c_2 \binom{1}{-1} \Rightarrow c_1 = \frac{4}{3},\ c_2 = \frac{2}{3}$

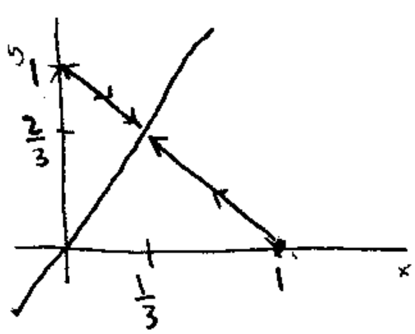
so $A^t \vec{e}_1 = A^t\left(\frac{4}{3}\binom{.25}{.5} + \frac{2}{3}\binom{1}{-1}\right) = \frac{4}{3}(1^t)\binom{.25}{.5} + \frac{2}{3}\left(\frac{1}{4}\right)^t \binom{1}{-1}$

$\Rightarrow \vec{x}(t) = \binom{\frac{1}{3}}{\frac{2}{3}} + \frac{2}{3}\left(\frac{1}{4}\right)^t \binom{1}{-1}$

$\vec{e}_2 = \binom{0}{1} = c_1 \binom{.25}{.5} + c_2 \binom{1}{-1} \Rightarrow c_1 = \frac{4}{3},\ c_2 = -\frac{1}{3}$

so $A^t \vec{e}_2 = A^t\left(\frac{4}{3}\binom{.25}{.5} - \frac{1}{3}\binom{1}{-1}\right) = \frac{4}{3}(1^t)\binom{.25}{.5} - \frac{1}{3}\left(\frac{1}{4}\right)^t \binom{1}{-1}$

$\Rightarrow \vec{x}(t) = \binom{\frac{1}{3}}{\frac{2}{3}} - \frac{1}{3}\left(\frac{1}{4}\right)^t \binom{1}{-1}$



b) $A^2 = \begin{bmatrix} .375 & .3125 \\ .625 & .6875 \end{bmatrix}$ $A^5 = \begin{bmatrix} .3340 & .3330 \\ .6660 & .6670 \end{bmatrix}$

$A^{10} = \begin{bmatrix} .3333 & .3333 \\ .6667 & .6667 \end{bmatrix}$ conjecture: $\lim\limits_{t\to\infty} A^t = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{2}{3} \end{bmatrix}$

This conjecture makes sense because eventually any point goes to some point on the line spanned by $\begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \end{pmatrix}$.

c) Write $\vec{e}_1$ and $\vec{e}_2$ in terms of eigenvectors:

$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = c_1 \begin{pmatrix} c \\ b \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \Rightarrow c_1 = \frac{1}{b+c} \quad c_2 = \frac{b}{b+c}$

$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = d_1 \begin{pmatrix} c \\ b \end{pmatrix} + d_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \Rightarrow d_1 = \frac{1}{b+c} \quad c_2 = \frac{-c}{b+c}$

1st column of A^t is $A^t \vec{e}_1 = A^t \left(\frac{1}{b+c} \begin{pmatrix} c \\ b \end{pmatrix} + \frac{b}{b+c} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right)$

$= \frac{1}{b+c} (1^t) \begin{pmatrix} c \\ b \end{pmatrix} + \frac{b}{b+c} (a-c)^t \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

as $t \to \infty$, the second term vanishes since $|a-c| < 1$.

Thus the first column approaches $\frac{1}{b+c} \begin{pmatrix} c \\ b \end{pmatrix}$.

Similarly, the second column is $A^t \vec{e}_2 = A^t \left(\frac{1}{b+c} \begin{pmatrix} c \\ b \end{pmatrix} - \frac{c}{b+c} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right)$

$= \frac{1}{b+c} (1^t) \begin{pmatrix} c \\ b \end{pmatrix} - \frac{c}{b+c} (a-c)^t \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{b+c} \begin{pmatrix} c \\ b \end{pmatrix}$ as $t \to \infty$.

Thus $\lim\limits_{t\to\infty} A^t = \frac{1}{b+c} \begin{bmatrix} c & c \\ b & b \end{bmatrix}$.

Math 21b
Homework #17 (TTh: HW #13)
Solutions by Dave Freeman

Sect. 6.2

22) $\det(\lambda I_n - A^T) = \det(\lambda I_n^T - A^T)$ since I_n is diagonal

$= \det((\lambda I_n - A)^T)$ since transpose is additive

$= \det(\lambda I_n - A)$ since determinant is unaffected by transpose.

Thus the characteristic polynomials of A and A^T are identical, and the two matrices have the same eigenvalues.

23) $\det(\lambda I_n - B) = \det(\lambda I_n - S^{-1}AS) = \det(\lambda S^{-1}S - S^{-1}AS)$

$= \det(S^{-1}(\lambda I_n - A)S) = \det(S^{-1})\det(\lambda I_n - A)\det S$

$= \det(\lambda I_n - A)$. Thus A and B have the same characteristic polynomial, so they have the same eigenvalues.

28) a)
$$\left.\begin{aligned} w(t+1) &= .8\,w(t) + .1\,m(t) \\ m(t+1) &= .2\,w(t) + .9\,m(t) \end{aligned}\right\} \Rightarrow \vec{x}(t+1) = \overbrace{\begin{bmatrix} .8 & .1 \\ .2 & .9 \end{bmatrix}}^{A} \vec{x}(t)$$

$A = \begin{bmatrix} .8 & .1 \\ .2 & .9 \end{bmatrix}$ is a regular transition matrix since each column sums to one.

b) From #25 (last time), we have
- eigenvector $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ with eigenvalue 1
- eigenvector $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ with eigenvalue .7

$\vec{x}(0) = \begin{pmatrix} 1200 \\ 0 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \Rightarrow c_1 = 400, \; c_2 = 800$

$\vec{x}(t) = A^t \vec{x}(0) = A^t \left(400 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + 800 \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right) = 400(1^t)\begin{pmatrix} 1 \\ 2 \end{pmatrix} + 800(.7^t)\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

so $w(t) = 400 + 800(.7^t)$
and $m(t) = 800 - 800(.7^t)$

c) The Wipfs will never have to shut down, as they will always have at least 400 customers.

Sect. 6.3

11) $\det \begin{bmatrix} \lambda-1 & -1 & -1 \\ -1 & \lambda-1 & -1 \\ -1 & -1 & \lambda-1 \end{bmatrix} = (\lambda-1)\left((\lambda-1)^2-1\right) + \left((-1)(\lambda-1)-1\right) - \left(1+(\lambda-1)\right)$

$= (\lambda-1)(\lambda^2-2\lambda) - \lambda - \lambda = \lambda^3 - 3\lambda^2 = \lambda^2(\lambda-3)$

eigenvalues are $0, 3$.

$\lambda = 3$: $\left[\begin{array}{ccc|c} 2 & -1 & -1 & 0 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 2 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 2 & -1 & -1 & 0 \\ 0 & 3 & -3 & 0 \\ 0 & -3 & 3 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 2 & -1 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$\longrightarrow \left[\begin{array}{ccc|c} 2 & 0 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

so the eigenspace is given by $t\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$; a basis is $\left\{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right\}$

$\lambda = 0$: $\left[\begin{array}{ccc|c} -1 & -1 & -1 & 0 \\ -1 & -1 & -1 & 0 \\ -1 & -1 & -1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

the eigenspace is given by $s\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$; a basis is $\left\{\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}\right\}$

The union of the two bases is an eigenbasis.

14) $\det \begin{bmatrix} \lambda-1 & 0 & 0 \\ 5 & \lambda & -2 \\ 0 & 0 & \lambda-1 \end{bmatrix} = \lambda(\lambda-1)(\lambda-1) \Rightarrow$ eigenvalues $0, 1$.

$\lambda = 0$: $\left[\begin{array}{ccc|c} -1 & 0 & 0 & 0 \\ -5 & 0 & -2 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

the eigenspace is $t\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$; a basis is $\left\{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right\}$.

$\lambda = \cancel{3}\,1$: $\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ -5 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & -\frac{1}{5} & \frac{2}{5} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

eigenspace is $s\begin{pmatrix} \frac{1}{5} \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -\frac{2}{5} \\ 0 \\ 1 \end{pmatrix}$; a basis is $\left\{\begin{pmatrix} 1 \\ 5 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 5 \end{pmatrix}\right\}$.

19) eigenspace $= \ker(\lambda I_n - A) = \ker \begin{bmatrix} 0 & -a & -b \\ 0 & 0 & -c \\ 0 & 0 & 0 \end{bmatrix}$

if $a=b=c=0$, $\ker(\lambda I_n - A) = \mathbb{R}^3$, so the geometric multiplicity is 3.

if $a \neq 0$ and $c \neq 0$, $\text{rank}(\lambda I_n - A) = 2$, so the kernel is 1-dimensional, and the geometric multiplicity is 1.

if one of a and c is zero, $\text{rank}(\lambda I_n - A) = 1$, so the kernel is 2-dimensional, and the geometric multiplicity is 2.

There is an eigenbasis only in the first case, when $A = I_3$.

20) $\lambda = 1$: eigenspace $= \ker \begin{bmatrix} 0 & -a & -b \\ 0 & 0 & -c \\ 0 & 0 & -1 \end{bmatrix} \xrightarrow{\text{"rref"}} \begin{bmatrix} 0 & a & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

the kernel will be 2-dimensional if $a=0$ and 1-dimensional otherwise; geometric multiplicity is 2 if $a=0$ and 1 otherwise.

$\lambda = 2$: eigenspace $= \ker \begin{bmatrix} 1 & -a & -b \\ 0 & 1 & -c \\ 0 & 0 & 0 \end{bmatrix}$. The rank of this matrix is 2,

so the kernel is 1-dimensional, so the geometric multiplicity is always 1.

A has an eigenbasis when $a=0$.

32) geometric multiplicity of λ as an eigenvalue of A^T is

$\dim(\ker(\lambda I_n - A^T)) = \dim(\ker((\lambda I_n - A)^T)) = n - \text{rank}((\lambda I_n - A)^T)$

$= n - \text{rank}(\lambda I_n - A)$ [since $\text{rank}(M^T) = \text{rank}\, M$]

$= \dim(\ker(\lambda I_n - A))$ = geometric multiplicity of λ as an eigenvalue of A.

34) The characteristic polynomial is a cubic with leading coefficient 1 and constant term $(-1)^3 \det A = -1$. Thus $f_A(0) = -1$ and $\lim\limits_{\lambda \to \infty} f_A(\lambda) = \infty$.

By the intermediate value theorem, $f_A(\lambda)$ must have a positive real root; by fact 6.1.2 this root can only be at $\lambda = 1$.

Thus A has an eigenvalue of 1, and the corresponding eigenvector is a fixed point under A.

43) a) $r(t+1) = .5\,r(t) + .25\,p(t)$

$p(t+1) = .5\,r(t) + .5\,p(t) + .5\,w(t)$

$w(t+1) = .25\,p(t) + .5\,w(t)$

So $$\begin{pmatrix} r(t+1) \\ p(t+1) \\ w(t+1) \end{pmatrix} = \begin{bmatrix} .5 & .25 & 0 \\ .5 & .5 & .5 \\ 0 & .25 & .5 \end{bmatrix} \begin{pmatrix} r(t) \\ p(t) \\ w(t) \end{pmatrix}$$

Find eigenvalues: $\det \begin{bmatrix} \lambda-.5 & -.25 & 0 \\ -.5 & \lambda-.5 & -.5 \\ 0 & -.25 & \lambda-.5 \end{bmatrix} = 0$; $(\lambda-.5)\left((\lambda-.5)^2 - .125\right) + .5\left(-.25(\lambda-.5)\right) = 0$

$$(\lambda-\tfrac{1}{2})(\lambda^2 - \lambda + \tfrac{1}{4} - \tfrac{1}{8}) + \tfrac{1}{2}(-\tfrac{1}{4}\lambda + \tfrac{1}{8}) = \lambda^3 - \tfrac{3}{2}\lambda^2 + \tfrac{1}{2}\lambda = \lambda(\lambda-1)(\lambda-\tfrac{1}{2}) = 0$$

Eigenvalues are $0, \frac{1}{2}, 1$.

Find eigenvectors:

$\lambda = 0$ $\begin{bmatrix} -\frac{1}{2} & -\frac{1}{4} & 0 \\ -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ 0 & -\frac{1}{4} & -\frac{1}{2} \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$; e-vect is $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$

$\lambda = \frac{1}{2}$ $\begin{bmatrix} 0 & -\frac{1}{4} & 0 \\ -\frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & -\frac{1}{4} & 0 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$; e-vect is $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

$\lambda = 1$ $\begin{bmatrix} \frac{1}{2} & -\frac{1}{4} & 0 \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & -\frac{1}{4} & \frac{1}{2} \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$; e-vect is $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$

$$\vec{x}_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + c_3 \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 & 1 \\ -2 & 0 & 2 \\ 1 & 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{4} \\ -\frac{1}{2} \\ \frac{1}{4} \end{pmatrix}$$

We thus have $\vec{x}(t) = A^t \vec{x}(0) = \frac{1}{4}(0^t)\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} - \frac{1}{2}\left(\frac{1}{2}\right)^t \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + \frac{1}{4}(1^t)\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$

So $$\begin{pmatrix} r(t) \\ p(t) \\ w(t) \end{pmatrix} = \begin{pmatrix} \left(\frac{1}{2}\right)^{t+1} + \frac{1}{4} \\ \frac{1}{2} \\ -\left(\frac{1}{2}\right)^{t+1} + \frac{1}{4} \end{pmatrix}$$ for $t > 0$.

b) As $t \to \infty$, $\begin{pmatrix} r(t) \\ p(t) \\ w(t) \end{pmatrix} \longrightarrow \begin{pmatrix} \frac{1}{4} \\ \frac{1}{2} \\ \frac{1}{4} \end{pmatrix}$ so the proportion is $1:2:1$.

44) a) $\begin{array}{l} a(t+1) = a(t) + j(t) \\ j(t+1) = a(t) \end{array} \Rightarrow \vec{x}(t+1) = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \vec{x}(t)$; $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

b) Find eigenvalues: $\det \begin{pmatrix} \lambda - 1 & -1 \\ -1 & \lambda \end{pmatrix} = 0$

$\Rightarrow \lambda(\lambda-1) - 1 = \lambda^2 - \lambda - 1 = 0$; $\lambda = \frac{1 \pm \sqrt{5}}{2}$

Find eigenvectors:

$\lambda = \frac{1+\sqrt{5}}{2}$ $\begin{bmatrix} -\frac{1+\sqrt{5}}{2} & -1 \\ -1 & \frac{1+\sqrt{5}}{2} \end{bmatrix} \updownarrow \text{swap} \rightarrow \begin{bmatrix} -1 & \frac{1+\sqrt{5}}{2} \\ -\frac{1+\sqrt{5}}{2} & -1 \end{bmatrix} \begin{array}{l} \times -1 \\ + \left(\frac{1-\sqrt{5}}{2}\right) I \end{array} \rightarrow \begin{bmatrix} 1 & \frac{-1-\sqrt{5}}{2} \\ 0 & 0 \end{bmatrix}$

so e-vect $= \begin{pmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{pmatrix}$

$\lambda = \frac{1-\sqrt{5}}{2}$ $\begin{bmatrix} -\frac{1-\sqrt{5}}{2} & -1 \\ -1 & \frac{1-\sqrt{5}}{2} \end{bmatrix} \updownarrow \text{swap} \rightarrow \begin{bmatrix} -1 & \frac{1-\sqrt{5}}{2} \\ -\frac{1-\sqrt{5}}{2} & -1 \end{bmatrix} \begin{array}{l} \times -1 \\ + \left(\frac{1+\sqrt{5}}{2}\right) I \end{array} \rightarrow \begin{bmatrix} 1 & -\frac{1+\sqrt{5}}{2} \\ 0 & 0 \end{bmatrix}$

so e-vect $= \begin{pmatrix} \frac{1-\sqrt{5}}{2} \\ 1 \end{pmatrix}$

$\vec{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = c_1 \begin{pmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} \frac{1-\sqrt{5}}{2} \\ 1 \end{pmatrix}$ $c_1 = -c_2$ and $(c_1 + c_2)\frac{1}{2} + (c_1 - c_2)\left(\frac{\sqrt{5}}{2}\right) = 1$

$\Rightarrow c_1 = \frac{1}{\sqrt{5}}, c_2 = \frac{-1}{\sqrt{5}}$

so $\begin{pmatrix} a(t) \\ j(t) \end{pmatrix} = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^t \begin{pmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{pmatrix} - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2}\right)^t \begin{pmatrix} \frac{1-\sqrt{5}}{2} \\ 1 \end{pmatrix}$

c) As $t \to \infty$, the second term above drops out as $\left| \frac{1-\sqrt{5}}{2} \right| < 1$.

Thus the limit of the ratio is simply the ~~com~~ ratio of the components of the ~~vector~~ eigenvector for $\lambda = \frac{1+\sqrt{5}}{2}$:

$$\lim_{t \to \infty} \frac{a(t)}{j(t)} = \frac{1+\sqrt{5}}{2}.$$

Sect. 6.4

20) $\det\begin{pmatrix} \lambda-3 & 5 \\ -2 & \lambda+3 \end{pmatrix} = (\lambda-3)(\lambda+3)+10 = \lambda^2+1 = 0$

$\Rightarrow \lambda = \pm i$

24) $\det\begin{pmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ -5 & 7 & \lambda-3 \end{pmatrix} = \lambda(\lambda(\lambda-3)+7) - 5(1) = \lambda^3-3\lambda^2+7\lambda-5=0$

rational roots theorem: only real roots can be $\pm 1, \pm 5$.

1 works:

$$\begin{array}{r|rrrr} 1 & 1 & -3 & 7 & -5 \\ & & 1 & -2 & 5 \\ \hline & 1 & -2 & 5 & 0 \end{array} \Rightarrow (\lambda-1)(\lambda^2-2\lambda+5)=0$$

eigenvalues are 1 and $\frac{2\pm\sqrt{4-20}}{2} = 1\pm 2i$

30) a) Write $\vec{x} = x_1\vec{e}_1 + \dots + x_n\vec{e}_n$, where $\sum x_i = 1$.

$A\vec{x} = A(x_1\vec{e}_1+\dots+x_n\vec{e}_n) = x_1A\vec{e}_1 + \dots + x_nA\vec{e}_n$.

The sum of the entries of $A\vec{x}$ is the sum of the entries of all the $x_iA\vec{e}_i$.

Since $A\vec{e}_i$ is the i^{th} column of A, the sum of its entries is 1.

Thus the sum of the entries of $A\vec{x}$ is the sum of the x_i, which is 1.

b) Using the example given, $A = \begin{bmatrix} .4 & .3 & .1 \\ .5 & .1 & .2 \\ .1 & .6 & .7 \end{bmatrix}$

$A^2 = \begin{bmatrix} .32 & .21 & .17 \\ .27 & .28 & .21 \\ .41 & .51 & .62 \end{bmatrix}$ $A^{10} \approx \begin{bmatrix} .21 & .21 & .21 \\ .24 & .24 & .24 \\ .55 & .55 & .55 \end{bmatrix}$

Conjecture: The columns of A^t always add up to 1, and they become the same as $n\to\infty$.

Proof: Let $\lambda_1, \dots, \lambda_n$ be the eigenvalues of A, with eigenvectors $\vec{v}_1, \dots, \vec{v}_n$.

Since there is a complex eigenbasis, we can write $\vec{e}_i = c_1\vec{v}_1 + \dots + c_n\vec{v}_n$.

$A^t\vec{e}_i = A^t(c_1\vec{v}_1+\dots+c_n\vec{v}_n) = c_1\lambda_1^t\vec{v}_1 + \dots + c_n\lambda_n^t\vec{v}_n$.

Since 1 is an eigenvalue of A with $\dim(E_1)=1$, we can assume $\lambda_1 = 1$.

All other eigenvalues have $|\lambda_i|<1$, so they go to zero as $n\to\infty$.

30 (cont'd)

thus $\lim_{t\to\infty} A^t \vec{e}_i = c_1\vec{v}_1$, so all columns of A^t are identical in the limit.

In addition, by part (a), since the entries of $\vec{e}_i$ sum to 1, the entries of $A^t\vec{e}_i = c_1\vec{v}_1$ must also sum to 1, so all columns of A^t sum to 1.

32) a)

$$\begin{aligned} a(t+1) &= .6\,a(t) + .1\,m(t) + .5\,s(t) \\ m(t+1) &= .2\,a(t) + .7\,m(t) + .1\,s(t) \\ s(t+1) &= .2\,a(t) + .2\,m(t) + .4\,s(t) \end{aligned} \qquad A = \begin{bmatrix} .6 & .1 & .5 \\ .2 & .7 & .1 \\ .2 & .2 & .4 \end{bmatrix}$$

b) Find the eigenvector for $\lambda = 1$: $\ker[I - A]$

$$\ker \begin{bmatrix} .4 & -.1 & -.5 \\ -.2 & .3 & -.1 \\ -.2 & -.2 & .6 \end{bmatrix} \begin{matrix} \times 10 \\ \times 20 \\ \times 20 \end{matrix} \longrightarrow \begin{bmatrix} 4 & -1 & -5 \\ -4 & 6 & -2 \\ -4 & -4 & 12 \end{bmatrix} \begin{matrix} \\ +I \\ +I \end{matrix} \longrightarrow \begin{bmatrix} 4 & -1 & -5 \\ 0 & 5 & -7 \\ 0 & -5 & 7 \end{bmatrix} \begin{matrix} \\ \div 5 \\ +II \end{matrix}$$

$$\longrightarrow \begin{bmatrix} 4 & -1 & -5 \\ 0 & 1 & -1.4 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} +II \\ \\ \\ \end{matrix} \longrightarrow \begin{bmatrix} 4 & 0 & -6.4 \\ 0 & 1 & -1.4 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} \div 4 \\ \\ \\ \end{matrix} \longrightarrow \begin{bmatrix} 1 & 0 & -1.6 \\ 0 & 1 & -1.4 \\ 0 & 0 & 0 \end{bmatrix}$$

e-vector is $\begin{pmatrix} 1.6 \\ 1.4 \\ 1 \end{pmatrix} = \begin{pmatrix} 8/5 \\ 7/5 \\ 1 \end{pmatrix}$

An eigenvector whose entries sum to 1 is $\begin{pmatrix} 2/5 \\ 7/20 \\ 1/4 \end{pmatrix} = \begin{pmatrix} .40 \\ .35 \\ .25 \end{pmatrix}$

From #30, we don't need to know the current market shares.

The shares in the long run are:

AT&T	40%
MCI	35%
Sprint	25%