HW # 19

6.4 / 36

6.5 / 5, 8, 12, 22, 28, 40afg, 42

6.4

36. a. Entries in the first row are age-specific birth rates
Entries immediately below diagonal are age-specific survival rates.

For example, during the next 15 years, 15-30 year olds will, on average, have 1.6 surviving child per person. 89% of people in this age bracket will themselves survive to become 30-45 years old 15 years in the future.

b. $\lambda_1 = 1.908$ $\quad \vec{v}_1 = \begin{bmatrix} .574 \\ .247 \\ .115 \\ .047 \\ .014 \\ .002 \end{bmatrix}$

$\vec{v}_1$ gives the long run population distribution

λ_1 gives the long run population growth rate.

annual growth factor $= \sqrt[15]{1.908} \approx 1.044$ or 4.4% growth

6.5

5. $A = \begin{bmatrix} .5 & .6 \\ -.3 & 1.4 \end{bmatrix}$

Note: This problem was not assigned

$\lambda_1 = 0.8$

$\lambda_2 = 1.1 > 1 \Rightarrow \vec{0}$ is not stable

8. $A = \begin{bmatrix} 1 & -0.2 \\ 0.1 & 0.7 \end{bmatrix}$

$\left.\begin{array}{l} \lambda_1 = 0.9 \\ \lambda_2 = 0.8 \end{array}\right\} < 1 \Rightarrow \vec{0}$ is stable state

12. $A = \begin{bmatrix} 0.6 & K \\ -K & 0.6 \end{bmatrix}$

$$\lambda_{1,2} = 0.6 \pm iK$$

stable iff $|\lambda_1| = |\lambda_2| = \sqrt{0.36 + K^2} < 1$

$$\Rightarrow K^2 < .64$$

$$\boxed{|K| < 0.8}$$

22. $A = \begin{bmatrix} 7 & -15 \\ 6 & -11 \end{bmatrix}$

Find e-vals:

$$\det \begin{bmatrix} \lambda-7 & 15 \\ -6 & \lambda+11 \end{bmatrix} = (\lambda-7)(\lambda+11) + 90 = \lambda^2 + 4\lambda + 13 = 0$$

$$\lambda = \frac{-4 \pm \sqrt{16-52}}{2}$$

$$\lambda = -2 \pm 3i, \quad r = \sqrt{13}, \quad \phi \simeq 2.16$$

Find e-vectors:

$$\begin{bmatrix} -9+3i & 15 \\ -6 & 9+3i \end{bmatrix} \xrightarrow{*(-9-3i)} \begin{bmatrix} 90 & -135-45i \\ -6 & 9+3i \end{bmatrix} \xrightarrow{\div 15} \begin{bmatrix} 6 & -9-3i \\ -6 & 9+3i \end{bmatrix} \rightarrow \begin{bmatrix} 2 & -3-i \\ 0 & 0 \end{bmatrix}$$

$$\vec{V}_1 = \begin{bmatrix} 3+i \\ 2 \end{bmatrix}$$

$$\vec{V}_{1,2} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \pm \begin{bmatrix} 1 \\ 0 \end{bmatrix} i$$

$$\vec{y} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \qquad c_1 \vec{y} + c_2 \vec{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow c_1 = -3/2, \quad c_2 = 1/2$$

$$x(t) = (\sqrt{13})^t \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \cos\phi t & -\sin\phi t \\ \sin\phi t & \cos\phi t \end{bmatrix} \begin{bmatrix} -3/2 \\ 1/2 \end{bmatrix}$$

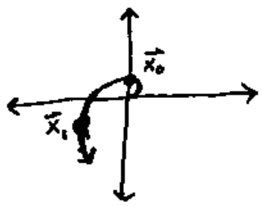
$$x(t) = (\sqrt{13})^t \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} -3/2 \cos\phi t - 1/2 \sin\phi t \\ -3/2 \sin\phi t + 1/2 \cos\phi t \end{bmatrix}$$

$$x(t) = \sqrt{13}^t \begin{bmatrix} -3/2 \cos\phi t - \cancel{1/2} \sin\phi t - 9/2 \sin\phi t + 3/2 \cos\phi t \\ -3\sin\phi t + \cos\phi t \end{bmatrix}$$

$$X(t) = (\sqrt{13})^t \begin{bmatrix} -5\sin\phi t \\ \cos\phi t - 3\sin\phi t \end{bmatrix} \quad \text{where } \phi \simeq 2.16$$

22. (continued)

$r = \sqrt{13} > 1 \Rightarrow$ spirals out



28. $\vec{x}(t+1) = (A - 2I_n)\vec{x}(t)$

Let λ be an eigen value of A

- $(\lambda - 2)$ is e-val of $(A - 2I_n)$

$|\lambda - 2| > 1$

$\Rightarrow$ not stable

40. a. $A^T A = (p^2 + q^2 + r^2 + s^2) I_4$

f. Let $A = \begin{bmatrix} 3 & -3 & -4 & -5 \\ 3 & 3 & 5 & -4 \\ 4 & -5 & 3 & 3 \\ 5 & 4 & -3 & 3 \end{bmatrix}$ and $\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 4 \\ 4 \end{bmatrix}$

then $A\vec{x} = \begin{bmatrix} -39 \\ 13 \\ 18 \\ 13 \end{bmatrix}$

$$39^2 + 13^2 + 18^2 + 13^2 = (3^2 + 3^2 + 4^2 + 5^2)(1^2 + 2^2 + 4^2 + 4^2) = 2183$$

g. Any positive integer can be expressed as $m = p_1 p_2 \cdots p_n$ using part f repeatedly, we see that

p_1, $p_1 p_2$, $p_1 p_2 p_3$, ..., $p_1 p_2 p_3 \cdots p_{n-1}$, and finally m can be expressed as the sum of four squares.

42. a. $A = \begin{bmatrix} 1 & -k \\ k & 1-k^2 \end{bmatrix}$

b. $f_A(\lambda) = \lambda^2 - (2-k^2)\lambda + 1 = 0$

The discriminant is $(2-k^2)^2 - 4 = -4k^2 + k^4 = k^2(k^2-4)$ is negative if k is a small positive number. Therefore, the eigen values are complex and the trajectory will be an elipse since $\det(A) = 1$.

HW #20

7.1/ 1, 3, 6, 8, 14, 21, 24, 28

1. $\begin{bmatrix} 7 \\ 16 \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ 5 \end{bmatrix} + c_2 \begin{bmatrix} 5 \\ 12 \end{bmatrix}$

$c_1 = -4$, $c_2 = 3$ is the solution, so

$\begin{bmatrix} -4 \\ 3 \end{bmatrix}$ is the desired coordinate vector

3. $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $S = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$

$S^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$

$B = S^{-1}AS = \begin{bmatrix} -1 & -1 \\ 4 & 6 \end{bmatrix}$

6. a. $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}$ are in the plane.

$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is perpendicular to the plane.

$B = \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ T(\vec{v}_1)_\beta & T(\vec{v}_2)_\beta & T(\vec{v}_3)_\beta \\ \downarrow & \downarrow & \downarrow \end{bmatrix}$

$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

b. $A = SBS^{-1}$

$= \begin{bmatrix} 1 & -1 & 1 \\ 1 & 2 & 2 \\ -1 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 1 & 2 & 2 \\ -1 & -1 & 3 \end{bmatrix}^{-1}$

$= \frac{1}{14} \begin{bmatrix} 12 & -4 & -6 \\ -4 & 6 & -12 \\ -6 & -12 & -4 \end{bmatrix}$

8. $B = \begin{bmatrix} 1 & 9 \\ 9 & 4 \end{bmatrix}$, $S = \begin{bmatrix} 3 & 5 \\ 5 & 8 \end{bmatrix}$

$S^{-1} = \begin{bmatrix} -8 & 5 \\ 5 & -3 \end{bmatrix}$

$A = SBS^{-1} = \begin{bmatrix} -74 & 54 \\ -111 & 82 \end{bmatrix}$

HW # 20 (page 2)

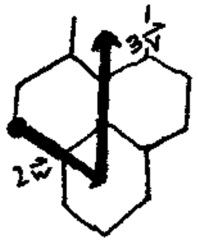
14. a. $\overrightarrow{OP} = \vec{w} + 2\vec{v}$

$\Rightarrow [\overrightarrow{OP}]_\beta = \begin{bmatrix}2\\1\end{bmatrix}$

$\overrightarrow{OQ} = \vec{v} + 2\vec{w}$

$\Rightarrow [\overrightarrow{OQ}]_\beta = \begin{bmatrix}1\\2\end{bmatrix}$

b. $\overrightarrow{OR} = 3\vec{v} + 2\vec{w}$



→ center of tile

c. observe that adding $3\vec{v}$ or $3\vec{w}$ to any vertex gives us another vertex.

Thus $17\vec{v} + 13\vec{w}$ is on a vertex iff $2\vec{v} + \vec{w}$ is on a vertex. From a) we know $2\vec{v} + \vec{w}$ is on a vertex

so S is on a vertex.

21. $T(\vec{v}_1) = \vec{v}_1 \times \vec{v}_2 = \vec{v}_3$ so $T(\vec{v}_1)_\beta = \begin{bmatrix}0\\0\\1\end{bmatrix}$

$T(\vec{v}_2) = \vec{v}_2 \times \vec{v}_2 = 0$ so $T(\vec{v}_2)_\beta = \begin{bmatrix}0\\0\\0\end{bmatrix}$

$T(\vec{v}_3) = \vec{v}_3 \times \vec{v}_2 = -\vec{v}_1$ so $T(\vec{v}_3)_\beta = \begin{bmatrix}-1\\0\\0\end{bmatrix}$

Thus $B = \begin{bmatrix}0 & 0 & -1\\0 & 0 & 0\\1 & 0 & 0\end{bmatrix}$

HW #20 (page 3)

24. a. $\vec{v}_0+\vec{v}_1+\vec{v}_2+\vec{v}_3 = \begin{bmatrix}1\\1\\1\end{bmatrix}+\begin{bmatrix}1\\-1\\-1\end{bmatrix}+\begin{bmatrix}-1\\1\\-1\end{bmatrix}+\begin{bmatrix}-1\\-1\\1\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$

b. $\vec{v}_0+\vec{v}_1+\vec{v}_2+\vec{v}_3 = \vec{0}$ (from above)

$\Rightarrow \vec{v}_0 = -\vec{v}_1-\vec{v}_2-\vec{v}_3$

$[\vec{v}_0]_\beta = \begin{bmatrix}-1\\-1\\-1\end{bmatrix}$

c. $T(\vec{v}_2) = T(-\vec{v}_0-\vec{v}_1-\vec{v}_3) = -\vec{v}_3-\vec{v}_0-\vec{v}_1 = \vec{v}_2$

Hence, T is $120°$ rotation around the line spanned by $\vec{v}_2$

$B = \left[\, T(\vec{v}_1)_\beta \;\; T(\vec{v}_2)_\beta \;\; T(\vec{v}_3)_\beta \,\right]$

$B = \begin{bmatrix}-1 & 0 & 1\\-1 & 1 & 0\\-1 & 0 & 0\end{bmatrix}$

$f_B(\lambda) = \lambda^3-1$, so $\lambda = 1, -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i = \cos(120°) \pm i\sin(120°)$

$B^3 = I_3$ since rotation by $120°$ three times returns a vector to its original position.

28. First, note that diagonal entries s_{ii} of S give the unit price of good i. If a_{ij} tells us how many dollars' worth of good i are required to produce one dollar's worth of good j, then $a_{ij}s_{jj}$ tells us how many dollars' worth of good i are required to produce one unit of good j, and $s_{ii}^{-1}a_{ij}s_{jj}$ is the number of units of good i required to produce on unit of good j. Thus $b_{ij} = s_{ii}^{-1}a_{ij}s_{jj}$, and

$$B = S^{-1}AS$$

$$B = \begin{bmatrix}.3 & .5 & .5\\.04 & .3 & .6\\.04 & .1 & .1\end{bmatrix}$$

HW # 21

7.2/ 14, 15, 21, 22, 23, 35, 54, 60

14. $f_A(\lambda) = \det \begin{bmatrix} \lambda+4 & -10 \\ 4 & \lambda-8 \end{bmatrix} = (\lambda+4)(\lambda-8)+40 = \lambda^2 - 4\lambda + 8$

$$\lambda = \frac{4 \pm \sqrt{16-32}}{2}$$

$$\lambda = 2 \pm 2i$$

$$E_{2+2i} = \operatorname{Ker} \begin{bmatrix} 6+2i & -10 \\ 4 & -6+2i \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 6+2i & -10 & 0 \\ 4 & -6+2i & 0 \end{array}\right] \xrightarrow{\cdot(6-2i)} \left[\begin{array}{cc|c} 40 & -60+20i & 0 \\ 4 & -6+2i & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 2 & -3+i & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$E_{2+2i} = \operatorname{span} \begin{bmatrix} 3-i \\ 2 \end{bmatrix}$$

$$E_{2-2i} = \operatorname{span} \begin{bmatrix} 3+i \\ 2 \end{bmatrix}$$

$$S = \begin{bmatrix} 3-i & 3+i \\ 2 & 2 \end{bmatrix}, \quad D = \begin{bmatrix} 2+2i & 0 \\ 0 & 2-2i \end{bmatrix}$$

$$S^{-1} \begin{bmatrix} -4 & 10 \\ -4 & 8 \end{bmatrix} S = D$$

15. $f_A(\lambda) = (\lambda+1)^2$, so $M = -A$ is a shear

Let $\vec{w}$ be some vector, not an eigenvector for M.

Then $\{\vec{w} - M\vec{w}, \vec{w}\}$ gives us a basis for $\mathbb{R}^2$

With respect to this basis, the matrix for M is

$$\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

If we let $\vec{w} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ then $\vec{w} - M\vec{w} = \begin{bmatrix} 8 \\ -4 \end{bmatrix}$, so $S = \begin{bmatrix} 8 & 0 \\ -4 & 1 \end{bmatrix}$ and $S^{-1}MS = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$

Then $S^{-1}AS = S^{-1}(-M)S = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$, as required

21. Yes.

let $S = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

then $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} = S^{-1} \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} S$

22. $A = \begin{bmatrix} -1 & 6 \\ -2 & 6 \end{bmatrix}$

$f_A(\lambda) = (\lambda+1)(\lambda-6)+12 = \lambda^2 - 5\lambda + 6 = (\lambda-2)(\lambda-3)$

$\lambda = 2, 3$

$\Rightarrow$ A is similar to $D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

$\boxed{\text{Yes}}$

23. Yes. Both shears are similar to $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ (see example 6 in textbook), so they are also similar to one another

35. $M = \frac{1}{2} A$ is a shear,

so $M^t(\vec{x}_0) = \vec{x}_0 + t(M\vec{x}_0 - \vec{x}_0)$ (see example 7 in textbook)

$= \begin{bmatrix} 0 \\ 1 \end{bmatrix} + t\left(\begin{bmatrix} -1/2 \\ 1/2 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -t/2 \\ 1 - t/2 \end{bmatrix}$

$\vec{x}(t) = A^t \vec{x}_0 = 2^t M^t \vec{x}_0 = 2^t \begin{bmatrix} -t/2 \\ 1 - t/2 \end{bmatrix} = 2^{t-1} \begin{bmatrix} -t \\ 2-t \end{bmatrix}$

54. First, observe that $\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}^t = \begin{bmatrix} 1 & at \\ 0 & 1 \end{bmatrix}$

Proof: (by induction)

$\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}^1 = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & a \cdot 1 \\ 0 & 1 \end{bmatrix}$

Next, if $\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}^{t-1} = \begin{bmatrix} 1 & (t-1)a \\ 0 & 1 \end{bmatrix}$ then

$\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}^t = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}^{(t-1)} \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & (t-1)a \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & ta \\ 0 & 1 \end{bmatrix}$

(continued)

54. (continued)

$$\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}^t = \lambda^t \begin{bmatrix} 1 & 1/\lambda \\ 0 & 1 \end{bmatrix}^t = \lambda^t \begin{bmatrix} 1 & t/\lambda \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \lambda^t & t\lambda^{t-1} \\ 0 & \lambda^t \end{bmatrix}$$

60. a. We are told that the dimension of the Eigenspace E_λ is d. We choose a basis $\vec{v}_1, \dots, \vec{v}_d$ of E_λ and a basis $\vec{v}_{d+1}, \dots, \vec{v}_n$ of $(E_\lambda)^\perp$, then $\vec{v}_1, \dots, \vec{v}_n$ will be a basis of $\mathbb{R}^n$.

b. For $i = 1, \dots, d$, (ith column of B) $= T(\vec{v}_i)_\beta = \begin{array}{cl} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \lambda_0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} & \begin{array}{l} \\ \\ \\ \leftarrow \text{ith component} \\ \\ \\ \\ \end{array} \end{array}$

so that $$B = \left[\begin{array}{c|c} \lambda_0 I_d & * \\ \hline 0 & * \end{array}\right]$$

c. $$f_B(\lambda) = \det \left[\begin{array}{c|c} (\lambda - \lambda_0) I_d & * \\ \hline 0 & * \end{array}\right] = (\lambda - \lambda_0)^d g(\lambda),$$ so that

algebraic multiplicity of λ_0 as e-val of A = alg. mult of λ_0 as e-val of B
$\geq d$
= geom. mult. of λ_0 as e-val of A.