HW #22

7.3//

2) We can easily see that $\begin{pmatrix}1\\1\end{pmatrix}$ & $\begin{pmatrix}-1\\1\end{pmatrix}$ are eigenvectors. As they are already orthogonal to each other, we just need to scale them to unit vectors.
Hence $\beta = \left\{ \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}-1\\1\end{pmatrix} \right\}$ is an orthonormal eigenbasis.

6) $\begin{pmatrix}2\\2\\1\end{pmatrix}, \begin{pmatrix}2\\-1\\-2\end{pmatrix}$, & $\begin{pmatrix}1\\-2\\2\end{pmatrix}$ are e-vectors, and they are orthogonal.

$\beta = \left\{ \frac{1}{3}\begin{pmatrix}2\\2\\1\end{pmatrix}, \frac{1}{3}\begin{pmatrix}2\\-1\\-2\end{pmatrix}, \frac{1}{3}\begin{pmatrix}1\\-2\\2\end{pmatrix} \right\}$ is an orthonormal eigenbasis.

10) $\lambda_1 = \lambda_2 = 0$ & $\lambda_3 = 9$.

$\vec{v}_1 = \frac{1}{\sqrt{5}}\begin{pmatrix}2\\1\\0\end{pmatrix} \in E_0$ & $\vec{v}_2 = \frac{1}{3}\begin{pmatrix}1\\-2\\2\end{pmatrix} \in E_9$

Let $\vec{v}_3 = \vec{v}_1 \times \vec{v}_2 = \frac{1}{3\sqrt{5}}\begin{pmatrix}2\\-4\\-5\end{pmatrix}$. Since A is symmetric, $\vec{v}_3$ is also an eigenvector. In fact $\vec{v}_3 \in E_0$. Hence,

$$S = \begin{pmatrix} 2/\sqrt{5} & 2/3\sqrt{5} & 1/3 \\ 1/\sqrt{5} & -4/3\sqrt{5} & -2/3 \\ 0 & -\sqrt{5}/3 & 2/3 \end{pmatrix} \text{ and } D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 9 \end{pmatrix}.$$

12) a) $E_1 = \text{span}\begin{pmatrix}1\\0\\2\end{pmatrix}$ and $E_{-1} = (E_1)^{\perp}$.

Choose $v_1 = \frac{1}{\sqrt{5}}\begin{pmatrix}1\\0\\2\end{pmatrix} \in E_1$ & $\vec{v}_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}$, $\vec{v}_3 = \frac{1}{\sqrt{5}}\begin{pmatrix}2\\0\\-1\end{pmatrix} \in E_{-1}$.

$\beta = \left\{ \frac{1}{\sqrt{5}}\begin{pmatrix}1\\0\\2\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \frac{1}{\sqrt{5}}\begin{pmatrix}2\\0\\-1\end{pmatrix} \right\}$ is an orthonormal eigenbasis.

b) $$B = \left([T(v_1)]_\beta, [T(v_2)]_\beta, [T(v_3)]_\beta \right) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

c) $$A = SBS^{-1} \text{ w/ } S = \begin{pmatrix} 1/\sqrt{5} & 0 & 2/\sqrt{5} \\ 0 & 1 & 0 \\ 2/\sqrt{5} & 0 & -1/\sqrt{5} \end{pmatrix}$$

$$\therefore A = \begin{pmatrix} -3/5 & 0 & 4/5 \\ 0 & -1 & 0 \\ 4/5 & 0 & 3/5 \end{pmatrix}$$

7.3//

16) a) A has 5 identical columns $\Rightarrow$ $\dim(\ker[A]) \geq 4$, and in fact we can easily see $\mathrm{Im}(A) \neq \{\vec{0}\}$, hence $\dim(\ker[A]) = 4$.

$\lambda_1 = 0$ has multiplicity 4.

Note $\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$ is an e-vector, w/ e-value 5, so $\lambda_2 = 5$ w/ multiplicity 1.

b) $B = A + 2I_5$

$\Rightarrow$ $\lambda_1 = 0+2 = 2$ w/ multiplicity 4

$\lambda_2 = 5+2 = 7$ w/ multiplicity 1.

c) $\det(B) =$ product of e-values $= 2^4 \cdot 7 = \boxed{112}$

26) J_n are both orthogonal and symmetric, so the eigenvalues are 1 & -1. As for their multiplicities, we know trace is the sum of e-values, and $\mathrm{tr}(J_n) = \begin{Bmatrix} 0 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd} \end{Bmatrix}$ & therefore

if n is even: 1 & -1 have multiplicity $n/2$

if n is odd: 1 has mult. $\frac{n+1}{2}$, -1 has mult. $\frac{n-1}{2}$.

36) Consider $\vec{v}$, an e-vector of A w/ e-value λ.

$\lambda\vec{v} = A\vec{v} = A^2\vec{v} = \lambda^2\vec{v}$.

$\Rightarrow \lambda = \lambda^2$

$\Rightarrow \lambda = 0$ or $\lambda = 1$.

Since A is symmetric E_0 & E_1 are orthogonal complements, so if $A \neq \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & 0 & \vdots \\ 0 & \cdots & 0 \end{pmatrix} = 0 \cdot I_n$ & $A \neq I_n$, A is projection onto a proper subspace E_1. If $A = I_n$, A is a "projection" onto $\mathbb{R}^2$ & if $A = 0 \cdot I_n$, it is a "projection" onto $\{\vec{0}\}$.

HW #23

7.4 ||

3) Let $A = \begin{pmatrix} 3 & 0 & 3 \\ 0 & 4 & 7/2 \\ 3 & 7/2 & 5 \end{pmatrix}$. Then $q(X) = X \cdot A\vec{X}$.

6) $A = \begin{pmatrix} 2 & 3 \\ 3 & 4 \end{pmatrix} \Rightarrow \det(A) < 0 \Rightarrow q$ is indefinite.

10) $L(X) = (X+V)^T A (X+V) - X^T A X - V^T A V$

$= X^T A X + X^T A V + V^T A X + V^T A V - X^T A X - V^T A V$

$= X^T A V + V^T A X$

But $X^T A V = (X^T A V)^T$ as $X^T A V$ is a scalar

$= V^T A^T X$

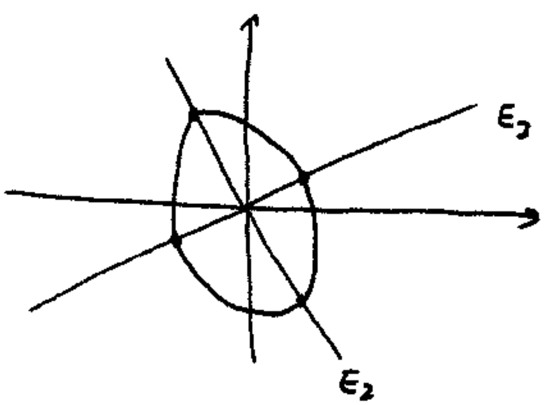
$= V^T A X$ as A is symmetric.

$\therefore\ L(X) = V^T A X + V^T A X$

$= (2V^T A)\vec{X}$. $L(X)$ is linear.

14) $\det(A)$ = product of the two eigenvalues. Hence, $\det(A) > 0$ implies that the two e-values have the same sign. Therefore, A is either positive definite or negative definite. Since $\vec{e_1} \cdot A\vec{e_1} = a > 0$, we know A is positive definite.

15) $A = \begin{pmatrix} 6 & 2 \\ 2 & 3 \end{pmatrix}$; e-values: $\lambda_1 = 7$, $\lambda_2 = 2$; $\beta = \left\{ \frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{5}}\begin{pmatrix} -1 \\ 2 \end{pmatrix} \right\}$ is an orthonormal eigenbasis.



$7C_1^2 + 2C_2^2 = 1$

7.4//

23) Yes. Let $A = \frac{1}{2}(M + M^T)$

Claim: A is symmetric

Pf: $A^T = \frac{1}{2}(M+M^T)^T = \frac{1}{2}(M^T + (M^T)^T) = \frac{1}{2}(M^T + M) = \frac{1}{2}(M + M^T) = A$.

Claim: $g(\bar{x}) = \bar{x} \cdot A\bar{x}$

~~[illegible]~~

Pf:

$$x^T A x = x^T \frac{1}{2}(M+M^T)x$$

$$= \frac{1}{2}(x^T M x + x^T M^T x)$$

But, $x^T M^T x$ is a scalar, so $x^T M^T x = (x^T M^T x)^T = x^T M x$

$$\Rightarrow x^T A x = \frac{1}{2}(x^T M x + x^T M x)$$

$$= x^T M x$$

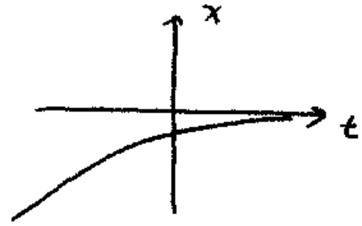
$$= g(x).$$

24) $q(\bar{e}_i) = \bar{e}_i \cdot A\bar{e}_i = \bar{e}_i \cdot (i^{th} \text{ column of } A) = a_{ii}$

HW #24

8.1//

2) By fact 8.1.1 $\quad x(t) = -e \cdot e^{-0.71t} = -e^{1-0.71t}$



14) a) By 8.1.1, $\quad x(t) = e^{-t/8270}$

Let T designate half-life. Then $e^{-T/8270} = 1/2$

$\Rightarrow \frac{-T}{8270} = \ln\left(\frac{1}{2}\right) \Rightarrow T = 8270 \cdot \ln 2 \approx 5732$

b) $e^{-t/8270} = 1 - 0.47 = 0.53$

$\Rightarrow t = -8270 \cdot \ln 0.53 \approx 5250$

The Iceman died 5,250 years before A.D. 1991, i.e about 3,250 B.C.

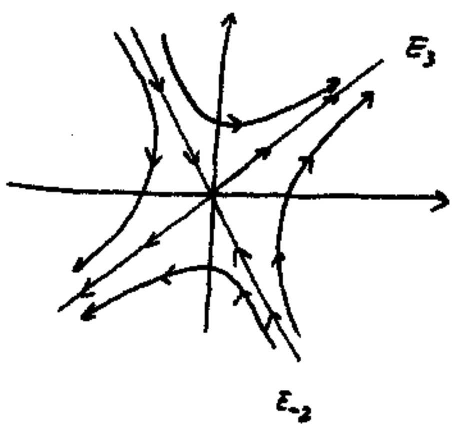
The Austrian expert was wrong.

26) $\lambda_1 = 3$; $\quad V_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$; $\quad c_1 = 5$

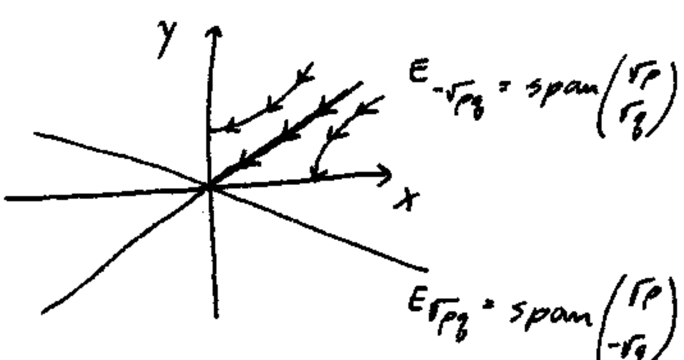
$\lambda_2 = -2$; $\quad V_2 = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$; $\quad c_2 = -1 \qquad \therefore \ \vec{x}(t) = 5 \cdot e^{3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} - e^{-2t}\begin{pmatrix} -2 \\ 3 \end{pmatrix}$.

32) $\lambda_1 = 3$

$\lambda_2 = -2$ from Ex. 26



46)



if $\frac{y(0)}{x(0)} < \frac{\sqrt{q}}{\sqrt{p}}$ $\quad x$ wins

if $\frac{y(0)}{x(0)} > \frac{\sqrt{q}}{\sqrt{p}}$ $\quad y$ wins

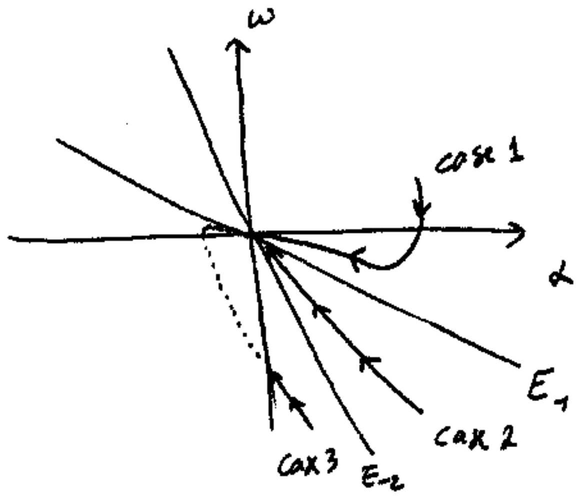
if $\frac{y(0)}{x(0)} = \frac{\sqrt{q}}{\sqrt{p}}$ $\quad$ everybody dies

HW # 24

8.1//

54) a) $A = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}$ $\lambda_1 = -1$ $V_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\lambda_2 = -2$ $V_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$



b) The door will slam only in case 3, i.e. if $\frac{\omega(0)}{\alpha(0)} < -2$.