Math 21b Final Exam Review - Spring 1999

Fall 1992 Final Exam

1) TRUE/FALSE

a) If 0 is an eigenvalue of the matrix **A**, then det(**A**)=0.

b) The set of solutions of $\frac{d^2x}{dt^2}+\frac{dx}{dt}-2x=0$ such that $\lim_{t\to\infty} x(t)=0$ forms a vector space of dimension 1.

c) If **A** is a 4×4 matrix, then det (-**A**) = -det(**A**).

d) The matrix $\begin{bmatrix} \cos q & -\sin q & 1 \\ \sin q & \cos q & 2 \\ 0 & 0 & 3 \end{bmatrix}$ is invertible for all q.

e) The vectors $\begin{bmatrix}1\\1\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\1\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\\1\end{bmatrix},\begin{bmatrix}1\\0\\0\\1\end{bmatrix}$ are linearly independent.

2) Consider the matrix **A** = $\begin{bmatrix} 0 & 2 & 0 & 0 \\ k & 0 & 2 & 0 \\ 0 & k & 0 & 2 \\ 0 & 0 & k & 0 \end{bmatrix}$ where k is constant.

(a) Find a value for the constant k such that the matrix **A** is diagonalizable. Justify your answer. (No partial credit will be given for guesses.)

(b) Find a value for the constant k such that the matrix **A** is not diagonalizable. Justify your answer.

3) Find the inverse of the matrix **A** = $\begin{bmatrix} 0 & 2 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{bmatrix}$.

4) Find the solution of the differential equation $\frac{d^2x}{dt^2}-3\frac{dx}{dt}+2x=0$ with x(0)=0, x(ln 2)=1.

5) A rabbit population and a wolf population are modeled by the equations

$$r(t+1)=5r(t)-2w(t)$$
$$w(t+1)=r(t)+2w(t)$$

The initial populations are r(0)=300 and w(0)=200.

(a) Find formulas for $r(t)$ and $w(t)$.

(b) In the long run, what will be the proportion of rabbits to wolves?

6) A rabbit population and a wolf population are modeled by the equations

$$\frac{dr}{dt}=4r-2w$$
$$\frac{dw}{dt}=r+w$$

The initial populations are r(0)=300 and w(0)=200.

(a) Find formulas for $r(t)$ and $w(t)$.

(b) In the long run, what will be the proportion of rabbits to wolves?

7) Consider the system

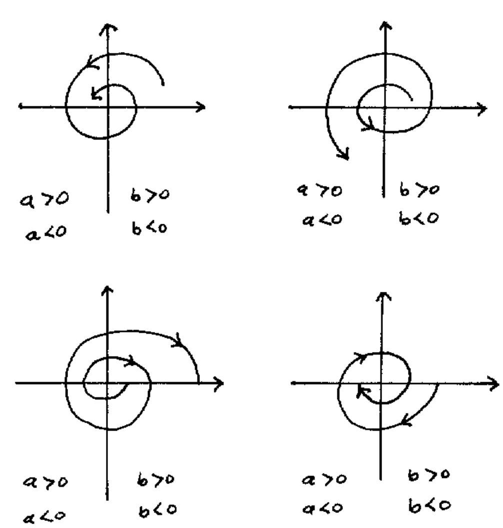
$$\frac{dx_1}{dt}=ax_1-bx_2$$
$$\frac{dx_2}{dt}=bx_1+ax_2$$

where a and b are constants.

(a) Sketch the phase portrait in the case a = 1, b = 0. Indicate clearly the shape of the trajectories (are they straight lines, circles, ellipses, etc.?).

(b) Sketch the phase portrait in the case a = 0, b = 1. Indicate clearly the shape of the trajectories.

(c) For each of the four phase portraits below, indicate which of the constants a and b are positive or negative (circle the correct answers).



8) Consider a herd of antelopes in Serengeti National Park. Here is a model for the instantaneous rate of change of the population of this herd:

- The birth rate is 10% per year.
- The rate of death from "natural causes" is 5% per year.
- The rate of predation (measured in animals per year) is proportional to the radical (i.e. the square root) of the population, with constant of proportionality 1.5 (this model makes sense when you consider the fact that predators [lions, etc.] will attack the herd at the periphery; think about it!).

(a) Set up a differential equation for the population P(t), reflecting the model presented above.
(b) Find the (positive) equilibrium solution for the differential equation you set up in (a), and determine its stability.
(c) Describe the long term behavior of the population and explain how this behavior depends on the initial population.

9) In the recent past, the real income of an average American family has doubled about every 30 years. However, this rate of growth has slowed, and at the current (percentage) rate it would take 200 years for the real family income to double. At this current rate, by how many percent does the income grow in a generation (in 30 years)?

10) Let V be the vector space of quadratic forms in two variables, i.e., functions of the form

$$f(x_1, x_2) = ax_1^2 + bx_1x_2 + cx_2^2.$$

Consider the linear transformation

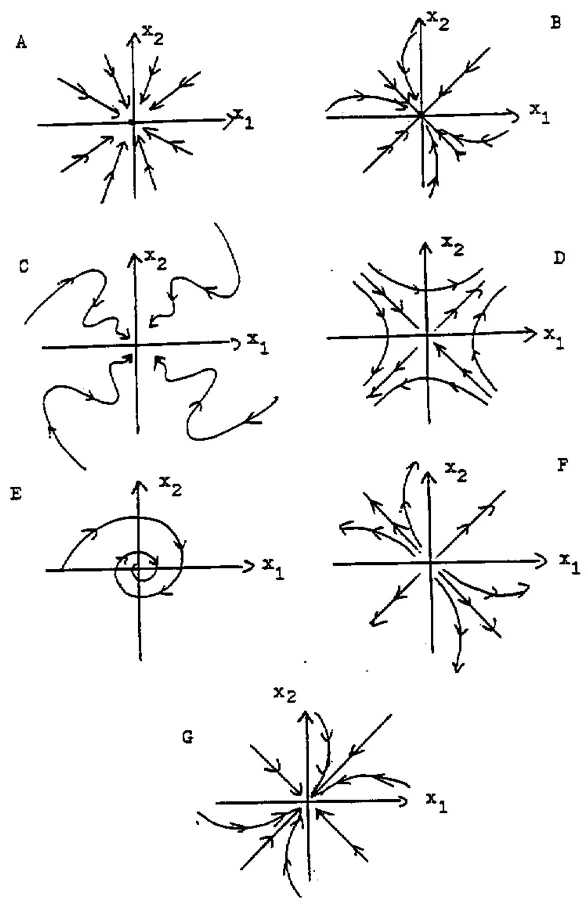
$$T : V \to V; \qquad T(f) = \frac{\partial f}{\partial x_1} \cdot x_2 - \frac{\partial f}{\partial x_2} \cdot x_1$$

(a) Find the matrix of the transformation T, with respect to the basis x_1^2, x_1x_2, x_2^2.
(b) Find bases for the kernel and image of T (note that kernel and image consist of elements of V, i.e., quadratic forms).

11) The <u>temperature</u> in the x_1x_2 plane is given by

$$T(x_1, x_2) = 30 - 3x_1^2 - 3x_2^2 - 2x_1x_2$$

(in degrees Celsius). Some heat-loving bugs crawl always in the direction in which the temperature increases most rapidly, i.e., in the direction of the <u>gradient</u> of T. Which of the sketches below best represents possible trajectories of these bugs? Justify your choice!



POSSIBLE TRAJECTORIES FOR PROBLEM 11

12) Consider the system $\left\{ \begin{array}{l} \dfrac{dx_1}{dt} = x_2^{\,2} - x_1 \\ \dfrac{dx_2}{dt} = x_1^{\,2} - x_2 \end{array} \right\}$.

(a) Perform the qualitative phase plane analysis for this system (i.e., find the null clines, equilibrium points, and general directions). Carry this out in the whole x_1-x_2 plane (not just the first quadrant).
(b) List the equilibrium points of the system above, and determine their stability.

13) Consider the vector space V consisting of all 2×2 matrices for which the vector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is an eigenvector. Find a basis for this space, and determine its dimension.

14) Consider the *van der Pol* equation

$$\frac{d^2x}{dt^2} = a(1 - x^2)\frac{dx}{dt} - x$$

where a is a <u>positive</u> constant.

(a) Transform the second order differential equation above into a system of two first order differential equations.
(b) Linearize your system, in (a), at the equilibrium point (0,0).
(c) Determine the stability of the origin; does your answer depend on the value of the positive constant *a*?
(d) For which choices of the constant *a* is the origin a node?

Fall 1993 Final Exam
TRUE/FALSE

(a) $x = 0$ is a stable equilibrium solution of the differential equation $\frac{dx}{dt} = x - x^3$.

(b) If **A** is an 8×5 matrix, then the kernel of **A** is at least three-dimensional.

(c) All trajectories of the dynamical system $\vec{x}(t+1) = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \vec{x}(t)$ approach (0,0) as *t* goes to infinity.

(d) There exist matrices **A** in $\mathbf{R}^{5\times4}$ and **B** in $\mathbf{R}^{4\times5}$ such that **AB** is invertible.

(e) Consider the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ in $\mathbf{R}^m$. Let **A** be a $p \times m$ matrix. If the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are linearly dependent, then so are the vectors $\mathbf{Av}_1, \mathbf{Av}_2, \ldots, \mathbf{Av}_n$.

(f) If **S** and **A** are orthogonal $n \times n$ matrices, then the matrix $\mathbf{S}^{-1}\mathbf{AS}$ is orthogonal as well.

(g) Orthogonal matrices are diagonalizable (over the reals).

(h) The differential equation $\frac{d^2x}{dt^2} + 7\frac{dx}{dt} + 13x = 0$ has no real solutions $x(t)$.

(i) Consider a diagonalizable 4×4 matrix **A**. If **A** is non-zero, then $\mathbf{A}^4$ is non-zero.

(j) Consider an invertible 4×4 matrix **A**. If all entries of **A** and of $\mathbf{A}^{-1}$ are integers, then det(**A**) = det ($\mathbf{A}^{-1}$).

1) Find an orthonormal eigenbasis for the matrix

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

2) Find all solutions of the system $\begin{Bmatrix} \frac{dx}{dt} = -8x - 6y \\ \frac{dy}{dt} = -x - 7y \end{Bmatrix}$ and sketch a phase portrait for this system.

3) Find the solution of the initial value problem

$$\frac{d^2x}{dt^2} - 6\frac{dx}{dt} + 25x = 0, \quad x(0) = 0, \quad x(\tfrac{p}{8}) = 1$$

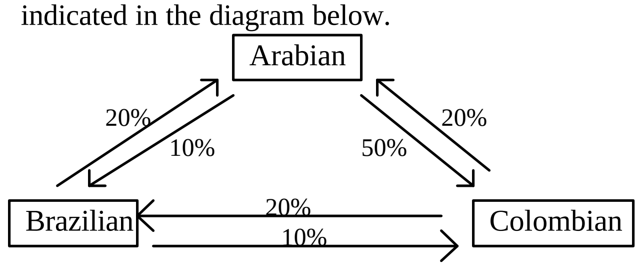
Draw a rough sketch of your solution.

4) "A certain person buys sheep, goats, and hogs, to the number of 100, for 100 crowns; the sheep cost him ½ a crown a-piece; the goats $1\frac{1}{3}$ crown; and the hogs 3½ crowns. How many had he of each?" (From Leonard Euler: Elements of Algebra, St. Petersburg, 1770. Translated by Rev. John Hewlett.)
Find all solutions to this problem.

5) Find a matrix **A** such that

$$\text{kernel}(\mathbf{A}) = \text{span}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \right).$$

6) A market research organization is studying a large group of coffee lovers who buy a bag of coffee each week. The choices are Arabian, Brazilian, and Colombian beans. Let $a(t)$, $b(t)$, and $c(t)$ be the fractions of the people in the sample who prefer the Arabian, the Brazilian, and the Colombian beans, *t* weeks after the beginning of the study. From one week to the next, people change their choice as indicated in the diagram below.



For example, 10% of the people who bought the Arabian beans a week ago will buy the Brazilian beans today, and 50% will switch to the Colombian beans, while the remaining 40% will again buy the Arabian beans.

(a) We introduce the state vector $\vec{x}(t) = \begin{bmatrix} a(t) \\ b(t) \\ c(t) \end{bmatrix}$.

Find the matrix **A** such that $\vec{x}(t+1) = \mathbf{A}\vec{x}(t)$. Time t is measured in weeks.

(b) We are told that $\left\{ \begin{bmatrix} 3 \\ 1 \\ -4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 5 \\ 7 \\ 8 \end{bmatrix} \right\}$ is an eigenbasis for **A**. In the long run, which fraction of the people in the sample will buy each of the types of beans?

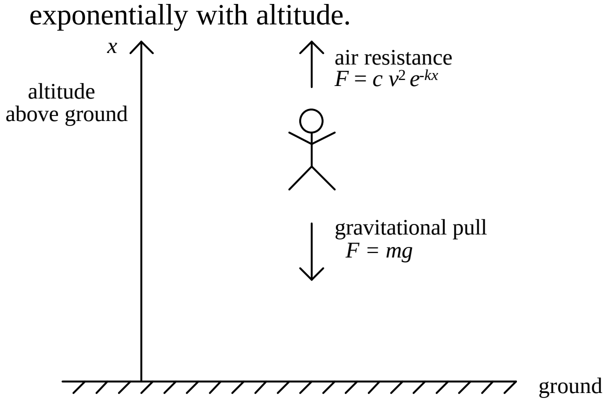
(c) Suppose that at the beginning of the study (at t = 0) everybody chooses the Arabian beans. Which fraction of the people in the study will choose the Arabian beans after t weeks?

7) Find a 2×2 matrix **A** with all of the following properties:

- det(**A**) = 2
- trace(**A**) = 3
- The vectors $\begin{bmatrix} 2 \\ 5 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ are eigenvectors of **A**.

8) Consider a skydiver falling vertically from a high altitude. Two forces are acting on her body:

- The force of gravity, mg, where m is her mass.
- The force of air resistance, which is assumed to be proportional to the square of her speed v, and also to the density of the air, which decreases exponentially with altitude.

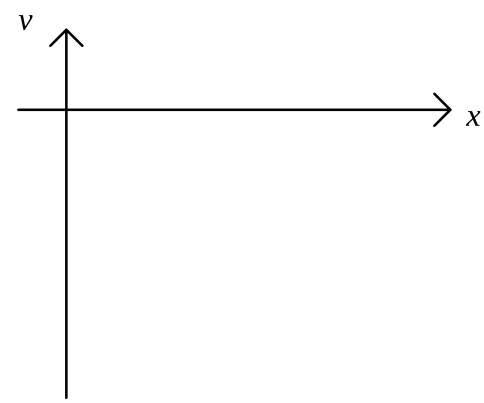


Note that the coordinates are chosen in such a way that the resistance force is positive and the gravitational force is negative.

(a) Set up a second order differential equation describing the motion of the skydiver. (Your formula will contain the constants c, k, m, and g.)

(b) Convert your solution, in (a), into a system of two first order differential equations.

(c) Perform the qualitative phase plane analysis on the system you found in (b). We are only interested in the case when x is positive and v is negative (the skydiver is falling!). Therefore, you need to consider the fourth quadrant only.



(d) Consider a trajectory starting at a point with v = 0. Relate the features of this trajectory to the fall of the skydiver.

9) Consider the matrix $\mathbf{A} = \begin{bmatrix} k & 0 & 0 & 0 & 1 \\ 0 & k & 0 & 0 & 1 \\ 0 & 0 & k & 0 & 1 \\ 0 & 0 & 0 & k & 1 \\ 1 & 1 & 1 & 1 & k \end{bmatrix}$,

where k is a constant.

(a) For which choice(s) of the constant k is the last column of **A** a linear combination of the first four columns?

(b) For which choice(s) of the constant k is the matrix **A** invertible?

(c) Consider the matrix $\mathbf{M} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}$.

Find as many linearly independent eigenvectors for **M** as you can.

10) The interaction of two species of animals is modeled by the differential equations

$$\begin{cases} \dfrac{1}{x}\dfrac{dx}{dt} = 1 - x + by - b \\ \dfrac{1}{y}\dfrac{dy}{dt} = 1 - y + bx - b \end{cases}$$

where b is a constant, different from 1 and -1.

(a) What does the sign of the constant b tell you about the way the two species interact?

(b) The system above has exactly one equilibrium point in the first quadrant, i.e. with $x > 0$ and $y > 0$. Find this equilibrium point.

(c) Linearize the system $\left\{ \begin{array}{l} \dfrac{dx}{dt} = x(1 - x + by - b) \\ \dfrac{dy}{dt} = y(1 - y + bx - b) \end{array} \right\}$ at the equilibrium point you found in (b).

(d) Determine the stability of the equilibrium point. Your answer will depend on the constant b.

Fall 1996 Final Exam

1) **True/False**. For full credit, briefly show your reasoning in each case.

(a) Let **A** be an $n \times (n - 1)$ matrix such that dim(ker **A**) = 1. Then for any **b** in $\mathbf{R}^n$, the solution set to **Ax** = **b** is a line parallel to ker **A**.

(b) Let **A** be an orthogonal matrix. Then det(**A**) = 1.

(c) If an $n \times n$ matrix **A** is diagonalizable, then it has n distinct eigenvalues.

(d) Let **A** be a symmetric matrix. Then ker($\mathbf{A}^2$) = ker(**A**).

(e) Let **A** be the matrix of an orthogonal projection. Then $\mathbf{A} = \mathbf{S}^T\mathbf{D}\mathbf{S}$ for some matrices **D** and **S**, where **D** is diagonal.

(f) Let $\mathbf{A} = \mathbf{S}^T\mathbf{D}\mathbf{S}$. Then **A** must be symmetric.

(g) $\det(\mathbf{A}^T\mathbf{A}^2\mathbf{A}^T) = -16$ for some matrix **A** in $\mathbf{R}^{7\times 7}$, the space of 7×7 matrices with real entries. (Here $\mathbf{A}^T$ denotes the transpose of **A**.)

(h) $\det \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = 0$.

(i) The function T: $\mathbf{R}^3 \to \mathbf{R}$ given by $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \det \begin{bmatrix} x_2 & 3 & 8 \\ x_1 & 4 & 2 \\ x_3 & 7 & 1 \end{bmatrix}$ is a linear transformation.

(j) If **A** is a real n x n matrix, where n is even, then there must be an even number of real eigenvalues.

2) Find the function of the form $f(t) = a + k \cos t$ best fitting the data points $(\pi/3, 1)$, $(2\pi/3, 0)$, $(\pi, 0)$ in the sense of least squares.

3) Let $E_\lambda(\mathbf{A})$ denote the eigenspace of **A** corresponding to the eigenvalue λ. Suppose V is a plane in $\mathbf{R}^3$ and that **A** and **B** are two different invertible matrices such that $E_{-2}(\mathbf{A}) = V$ and $E_{-2}(\mathbf{B}) = V$.

a) Show that any vector in V is in $E_{-2}(6\mathbf{A} - 4\mathbf{B})$.

b) Show that any vector in V is in $E_1(2\mathbf{A}^3\mathbf{B}^{-4})$. (Note: $\mathbf{B}^{-4}$ denotes $(\mathbf{B}^{-1})^4$ or $(\mathbf{B}^4)^{-1}$.)

c) If **P** is orthogonal projection onto V, what can you say about $E_{-2}(\mathbf{AP})$?

4) Suppose **A** is a symmetric 5×5 matrix and $\mathbf{A}^5 = \mathbf{A}^3$. Assume that det(**A**) < 0 and trace(**A**) = 3.

(a) Determine the eigenvalues of **A** and their algebraic and geometric multiplicities.

(b) What are the dimensions of the kernel and image of **A**?

(c) Describe what **A** does geometrically. Be as specific as possible.

5) Let $\mathbf{x} \in \mathbf{R}^4$ and set $M = \begin{bmatrix} 1 & 3 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}$. Define $T(\mathbf{x}) = \begin{bmatrix} M_{12} \\ M_{21} \end{bmatrix}$, where M_{ij} = (i,j) entry of matrix M. T is a linear transformation.

(a) Find the matrix of T with respect to the standard bases for its domain and range.

(b) Find an orthonormal basis for kernel(T).

(c) Is the set kernel(T) isomorphic to P_2 = {polynomials in t of degree ≤ 2}?

6) Consider the quadratic form $q(\mathbf{x}) = x_1^2 - 6x_1x_2 + x_2^2$, where $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

(a) Find a symmetric matrix **A** such that $q(\mathbf{x}) = \mathbf{x}^T\mathbf{A}\mathbf{x}$.

(b) Find an orthonormal basis B and a diagonal matrix **D** such that the matrix of q with respect to the basis B is **D**.

(c) In $\mathbf{R}^2$ with standard coordinates (x_1, x_2), sketch the set of solutions of the equation $q(\mathbf{x}) = 2$.

(d) Find the standard coordinates of a nonzero vector **y** such that $q(\mathbf{y}) = 0$.

7) Let **A** be a 2×2 matrix. Consider the continuous dynamical system $\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x}$.

In parts (a) - (d) below, consider each situation separately and justify your answers carefully.

(a) If **A** is symmetric and $\mathbf{x}(0) \neq \mathbf{0}$, what are the minimum and maximum number of quadrants that a trajectory $\mathbf{x}(t)$ can intersect?
(Here quadrant refers to the standard four quadrants of the xy-plane.

(b) If **A** has a complex eigenvalue and $\mathbf{x}(0) \neq \mathbf{0}$, what are the minimum and maximum number of quadrants that a trajectory $\mathbf{x}(t)$ can enter?

(c) Suppose $\lim_{t\to\infty} x_1(t) = 0$ for all trajectories of the system. Assume **A** is symmetric. What can be said about the eigenvalues and eigenvectors of **A**?

(d) Suppose some trajectory of this system has the property that $\lim_{t\to\infty} \frac{x_2(t)}{x_1(t)} = 3$. Find an eigenvector of **A**. Must all trajectories have this limit?

8) Let P_n be the space of polynomial functions spanned by the basis B = $\{1, t, t^2, ..., t^n\}$. Let Q_n be the space of functions spanned by the basis C = $\{f_0, f_1, f_2, ..., f_n\}$ where $f_0 = 1$, $f_1 = t + 1$, $f_2 = (t + 1)(t + 2)$, etcetera.

a) Express $2f_2 + 3f_1 + f_0$ in terms of the basis B.

b) Express 1, t, and t^2 in terms of the basis C.

c) Express $2t^2 + 3t + 1$ in terms of the basis C.

d) Compute the change of basis matrix **S** from P_2 to Q_2, i.e. the matrix **S** such that $[f]_B = \mathbf{S}[f]_C$.

9) Let V be the space of functions defined on $[-\pi, \pi]$ which are continuous except for a finite number of points, with inner product given by:

$$< f, g > = \tfrac{1}{\pi}\int_{-\pi}^{\pi} f(t)g(t)dt$$

Let h in V be the function defined by:

$$h(t) = \begin{Bmatrix} 0 & \text{if} & t \in [-\pi, -\pi/3) \\ 1 & \text{if} & t \in [-\pi/3, \pi/3] \\ 0 & \text{if} & t \in (\pi/3, \pi] \end{Bmatrix}$$

Let T_3 be the subspace of V spanned by the basis $\{ 1/\sqrt{2}, \cos(t), \sin(t), \cos(2t), \sin(2t), \cos(3t), \sin(3t)\}$.

Find the orthogonal projection of h onto T_3.

10) (a) Find all solutions of the differential equation

$$f'' - f' - 6f = 6t.$$

(b) Find a solution which satisfies $f'(0) = f''(0) = 30$. Is such a solution unique?

Spring 1997 Final Exam

1) TRUE/FALSE

(a) If the eigenvalues for a 2×2 matrix are $\frac{11}{12}$ and $\frac{13}{12}$, and $\vec{v}$ is neither **0** nor an eigenvector for **A**, then $\|\mathbf{A}^n\vec{v}\|$ will go to ∞ as $n \to \infty$.

(b) If **A** is a real 2×2 matrix and det(**A**) < 0, then all eigenvalues of **A** are real.

(c) If $T: V \to V$ is a linear transformation such that $T^2 = T$, then dim(Ker T) = dim(Range T).

(d) If a matrix **A** is diagonalizable with $\mathbf{A} = \mathbf{SDS}^{-1}$, and **A** is invertible, then $\mathbf{A}^{-1} = \mathbf{SD}^{-1}\mathbf{S}^{-1}$.

(e) $\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$ has 5 linearly independent eigenvectors.

(f) $\begin{Bmatrix} \frac{dx}{dt} = x + 2y \\ \frac{dy}{dt} = -4x + y \end{Bmatrix}$ has no real solutions.

(g) If λ is an eigenvalue of **A** with eigenvector **v**, μ is a distinct eigenvalue with eigenvector **w**, then **v** + **w** is also an eigenvector of **A**.

(h) If T_1, T_2 are two linear transformations from $\mathbf{R}^3$ to $\mathbf{R}^3$ such that Ker(T_1) = Ker(T_2) and Range(T_1) = Range(T_2), then $T_1 = T_2$.

(i) If 0 is an eigenvalue of **A**, then the system of linear equations **Ax** = **0** has infinitely many solutions.

(j) The solutions to the differential equation $y'' + 4y' + 2y = \sin x$ form a 2-dimensional vector space.

2) (10 points) Let $\mathbf{A} = \begin{bmatrix} 1 & 0 & 3 & -1 \\ 2 & 1 & 5 & 1 \\ 2 & 0 & 3 & -1 \\ 1 & 1 & 2 & 2 \end{bmatrix}$.

(a) Find bases for Kernel(**A**) and Range(**A**). What are their dimensions?

(b) Find det(**A**).

3) (12 points) Let $\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & t \\ 1 & 0 & 1 & t^2 \\ -1 & 0 & 1 & t^3 \end{bmatrix}$.

(a) Calculate det(**A**) as a function of t.

(b) Determine the kernel of **A** for every t.

4) (12 points) (a) Find the general solution to the differential equation $y''(x) - 4y'(x) + 13y(x) = 0$.

(b) Find the solution to the equation in part (a) which satisfies $y(0) = 1$ and $y'(0) = 2$.

5) (12 points) (a) Let $\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$. Is **A** diagonalizable? Diagonalize **A** if your answer is yes; explain why if the answer is no.

(b) Is there an orthogonal matrix **S** such that $\mathbf{S}^{-1}\mathbf{AS}$ is diagonal? Find such an **S** if your answer is yes; explain why if the answer is no.

6) Consider an entering class of 1200 students in a two-year community college. After t years each of these students will belong to one of the following groups:

- Will still be a first year student
- Will be a second year student
- Will have graduated
- Will have dropped out (we do not consider students who drop out and rejoin the class later)

Let $f(t)$, $s(t)$, $g(t)$, and $d(t)$ be the numkber of students in each of these groups, after t years.

Consider the state vector $\mathbf{x}(t) = \begin{bmatrix} f(t) \\ s(t) \\ g(t) \\ d(t) \end{bmatrix}$;

we were told that $\mathbf{x}(0) = \begin{bmatrix} 1200 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

We are told that $\mathbf{x}(t+1) = \mathbf{Ax}(t)$, with

$\mathbf{A} = \begin{bmatrix} 1/4 & 0 & 0 & 0 \\ 1/2 & 1/5 & 0 & 0 \\ 0 & 3/5 & 1 & 0 \\ 1/4 & 1/5 & 0 & 1 \end{bmatrix}$.

(a) What is the meaning of the entry 3/5 in this matrix?

(b) Find all eigenvalues of this matrix.

(c) You are told that $\begin{bmatrix} 1 \\ 10 \\ -8 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ -3 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ is an eigenbasis of **A**, and that

$$\begin{bmatrix} 1200 \\ 0 \\ 0 \\ 0 \end{bmatrix} = 1200\begin{bmatrix} 1 \\ 10 \\ -8 \\ -3 \end{bmatrix} - 3000\begin{bmatrix} 0 \\ 4 \\ -3 \\ -1 \end{bmatrix} + 600\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + 600\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

How many of the 1200 students entering the class will eventually graduate?

(7) Consider the transformation T: $\mathbf{R}^{2\times 2} \to \mathbf{R}^{2\times 2}$ given by $T(\mathbf{A}) = \mathbf{A}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\mathbf{A}$.

(a) Verify that T is a linear transformation.

(b) Calculate $T(\begin{bmatrix} a & b \\ c & d \end{bmatrix})$ and find the matrix of T with respect to the standard basis of $\mathbf{R}^{2\times 2}$.

(c) Find bases of the kernel and image of T.

(d) Is T invertible?

8) Match each system with the phase portrait shown which best represents it.

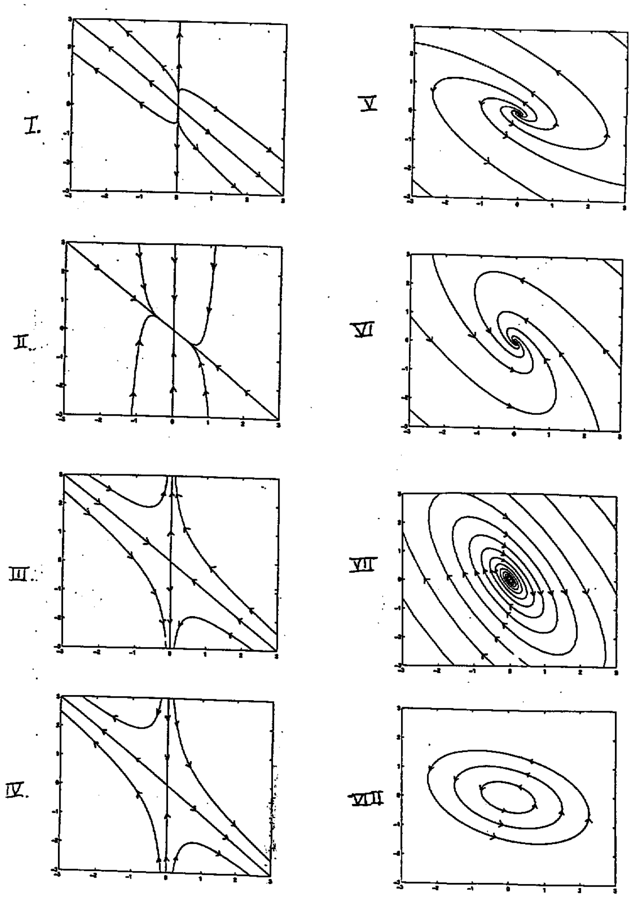
(a) $\mathbf{x}(t+1) = \begin{bmatrix} 3 & 0 \\ -2.5 & 0.5 \end{bmatrix}\mathbf{x}(t)$

(b) $\mathbf{x}(t+1) = \begin{bmatrix} -1.5 & -1 \\ 2 & 0.5 \end{bmatrix}\mathbf{x}(t)$

(c) $\dfrac{d\mathbf{x}}{dt} = \begin{bmatrix} 3 & 0 \\ -2.5 & 0.5 \end{bmatrix}\mathbf{x}$

(d) $\dfrac{d\mathbf{x}}{dt} = \begin{bmatrix} -1.5 & -1 \\ 2 & 0.5 \end{bmatrix}\mathbf{x}$

(e) $\dfrac{d\mathbf{x}}{dt} = \begin{bmatrix} -2 & 0 \\ 3 & 1 \end{bmatrix}\mathbf{x}$



9) Consider a linear transformation T from $\mathbf{R}^n$ to $\mathbf{R}^m$. Explain why there are orthonormal vectors $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$ in $\mathbf{R}^n$ such that the vectors $T(\mathbf{v}_1), T(\mathbf{v}_2), \cdots, T(\mathbf{v}_n)$ are orthogonal.

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1) (20 points) TRUE/FALSE

(a) Any symmetric 2×2 matrix has two distinct eigenvalues.

(b) Any symmetric 2×2 matrix satisfying $\mathbf{A}^2 = \mathbf{I}_2$ is positive definite.

(c) If $\mathbf{A}$ is an invertible matrix that is similar to $\mathbf{A}^{-1}$, then det($\mathbf{A}$) is either +1 or -1.

(d) Suppose that $\mathbf{A}$ and $\mathbf{B}$ are simultaneously diagonalizable n × n matrices, i.e. there is an invertible matrix $\mathbf{S}$ such that $\mathbf{S}^{-1}\mathbf{AS}$ and $\mathbf{S}^{-1}\mathbf{BS}$ are both diagonal. Then $\mathbf{AB}$ must equal $\mathbf{BA}$.

(e) Suppose that $\mathbf{A}$ and $\mathbf{B}$ are two diagonalizable n × n matrices. Then $\mathbf{AB}$ must equal $\mathbf{BA}$.

(f) If $\mathbf{A}$ is a 2×2 matrix with Tr($\mathbf{A}$) = 0 and det ($\mathbf{A}$) =1, then $\mathbf{A}^4 = \mathbf{I}_2$.

(g) Let V be the space of (real) solutions of the differential equation $f''(t) + 7f'(t) + 10f(t) = 0$. The linear transformation $T(f(t)) = f'(t)$ from V to V is diagonalizable.

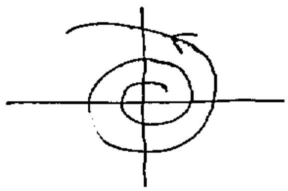
(h) If two matrices have the same characteristic polynomial, then they have the same rank.

(i) Let $\mathbf{v} = \begin{bmatrix} 1 \\ 9 \\ 9 \\ 8 \end{bmatrix}$. If $\mathbf{A}$ and $\mathbf{B}$ are similar 4×4 matrices, and $\mathbf{Av} = \mathbf{0}$, than $\mathbf{Bv} = \mathbf{0}$ as well.

(j) There is a real 2×2 matrix $\mathbf{A}$ that has $\begin{bmatrix} 1 \\ 9 \end{bmatrix}$ as an eigenvector, with associated eigenvector $9 + 8i$.

2) (8 points) Consider the linear system $\frac{d\vec{x}}{dt} = \begin{bmatrix} 0 & -1 \\ k & 2k \end{bmatrix}\vec{x}$, where k is an arbitrary real number. For each phase portrait below, find a possible value of k such that the phase portrait of the system is of the pictured type, or state that no such k exists.

(a)

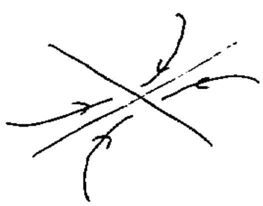


$k =$

(b)

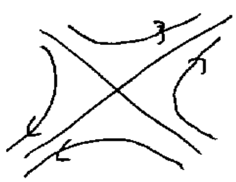
$k =$

(c)



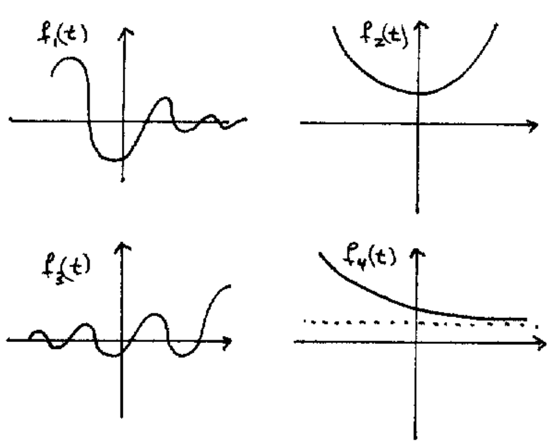
$k =$

(d)



$k =$

3) (8 points) Below are the graphs of five different functions:



Each of these functions satisfies precisely one of the following five differential equations. Write the number of each function to the left of the differential equation it satisfies.

( ) $f'' - 2f' + 401f = 0$

( ) $f'' + 2f' + 401f = 0$

( ) $f'' - f = 0$

( ) $f'' + f = 0$

4) (10 points) (a) Let P be the linear space of all polynomials. Consider the linear transformation $T{:}P \to P$ given by $T(f(t)) = t\,f'(t)$. Find the eigenvalues and eigenfunctions of T.

(b) Let V be the linear space of infinite sequences $(a_0, a_1, a_2, a_3,\ldots.)$ of real numbers. Give an example of a linear transformation $T{:}V \to V$ whose eigenvalues are precisely all the non-negative integers 0,1,2,3,....

5) (10 points) Let $\mathfrak{B}$ be the basis of $\mathbf{R}^3$ consisting of the vectors $\mathbf{v}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix}3\\1\\0\end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix}1\\0\\1\end{bmatrix}$.

Let $T{:}\ \mathbf{R}^3 \to \mathbf{R}^3$ be the linear transformation that swaps $\mathbf{v}_1$ with $\mathbf{v}_2$ and sends $\mathbf{v}_3$ to $-\mathbf{v}_3$.

(a) Find the matrix $\mathbf{B}$ of T with respect to the basis $\mathfrak{B}$.

(b) Find the matrix $\mathbf{A}$ of T with respect to the standard basis $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ of $\mathbf{R}^3$.

6) (10 points)
Let $q(x_1, x_2) = -7x_1^2 + 48x_1x_2 + 7x_2^2 = \mathbf{x}^T\mathbf{A}\mathbf{x}$, where $\mathbf{A}$ is a symmetric 2×2 matrix.

(a) Does $\mathbf{R}^2$ have an orthonormal eigenbasis with respect to $\mathbf{A}$? If so, find one. If not, why not?

(b) Sketch the curve $q(x_1, x_2) = 25$.

7) (12 points) Consider the system of couple differential equations $\left\{\begin{array}{l} \frac{dx}{dt} = -x + (2-x)y \\ \frac{dy}{dt} = -y + (2-y)x \end{array}\right\}$. We are only interested in what happens in the closed

region $0 \le x,y \le 2$.
There is one equilibrium point (a, b) in the *interior* of the region, i.e. $0 < a,b < 2$. The linearization of the system at (a, b) has the form $\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \mathbf{J}\begin{bmatrix} x-a \\ y-b \end{bmatrix}$.

(a) Find (a, b) and $\mathbf{J}$.

(b) Find an eigenbasis of $\mathbf{R}^2$ with respect to $\mathbf{J}$, together with the corresponding eigenvalues. What does this tell you about the stability (or lack thereof) of the point of equilibrium (a, b)?

(c) Sketch the horizontal and vertical nullclines (the curves where the tangent vectors are horizontal and vertical, respectively). Be sure to indicate all equilibrium points in the region $0 \le x, y \le 2$. For each subregion cut out by the nullclines, indicate (using a single arrow) the direction of flow of the system in that region.

(d) State whether or not each of the equilibrium points in the closed region $0 \le x, y \le 2$ is a point of stable equilibrium.

9) (10 points) Let T: $\mathbf{R}^{2\times2} \to \mathbf{R}^{2\times2}$ be the transformation $T(\mathbf{A}) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\mathbf{A} - \mathbf{A}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

(a) Show that T is a linear transformation.

(b) Find the matrix $\mathbf{M}$ of T with respect to the standard basis $\left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}$ of $\mathbf{R}^{2\times2}$.

(c) Find the kernel and image of T. (You may do this using $\mathbf{M}$, or directly from the definition of T.)

(d) Does $\mathbf{R}^{2\times2}$ have an eigenbasis with respect to T?

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1) TRUE/FALSE

(a) If $\mathbf{A}$ is a 3×3 orthogonal matrix, then $\det(\mathbf{A})$ is an eigenvalue of $\mathbf{A}$.

(b) If T is a linear transformation from $\mathbf{R}^n$ to $\mathbf{R}^n$ which sends orthogonal vectors to orthogonal vectors, then T is an orthogonal transformation.

(c) If $\mathbf{A}$ is a real 4×4 matrix with determinant 1/2 and with no real eigenvalue, then $\mathbf{A}^n \to \mathbf{0}$ as $n \to \infty$

(d) Suppose that $\mathbf{A}$ is a 2×2 real matrix and that the discrete dynamical system $\mathbf{x}(t + 1) = \mathbf{A}\mathbf{x}(t)$ has a non-constant solution satisfying $\mathbf{x}(t + 4) = \mathbf{x}(t)$, then $\mathbf{0}$ is an unstable equilibrium solution of $\mathbf{x}(t + 1) = \mathbf{A}\mathbf{x}(t)$.

(e) If there are orthonormal eigenbases of $\mathbf{R}^n$ for each of two matrices $\mathbf{A}$ and $\mathbf{B}$, then there is also one for $\mathbf{A} + \mathbf{B}$.

(f) The solutions to $\frac{d^2 f}{dt^2} + \frac{df}{dt} + f = e^t$ form a two dimensional linear subspace of C^∞.

2) (a) Find the eigenvalues and eigenvectors for $\begin{bmatrix} 3 & 2 \\ 10 & 4 \end{bmatrix}$.

(b) Find a matrix $\mathbf{A}$ such that $\mathbf{A}^3 = \begin{bmatrix} 3 & 2 \\ 10 & 4 \end{bmatrix}$.

3) Suppose that $\mathbf{A}$ is a 3×5 matrix and $\mathbf{B}$ is a 5×3 matrix such that $\mathbf{AB} = \mathbf{I}_3$.

(a) Show that the non-zero vectors in Ker($\mathbf{A}$) are eigenvectors for $\mathbf{BA}$. What is their eigenvalue?

(b) Show that the non-zero vectors in Im($\mathbf{B}$) are eigenvectors for $\mathbf{BA}$. What is their eigenvalue?

(c) List all the eigenvalues of $\mathbf{BA}$ with their geometric multiplicities. Is there an eigenbasis of $\mathbf{R}^5$ for $\mathbf{BA}$?

4) Let $\mathbf{v}_1 = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$.

Match the following matrices with the verbal descriptions below.

(a) $\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ (b) $\mathbf{B} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

(c) $\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$ (d) $\mathbf{D} = \begin{bmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \end{bmatrix}$

(i) The matrix, with respect to the standard basis of $\mathbf{R}^3$, for reflection in the line spanned by $\mathbf{v}_1$.

(ii) The matrix with respect to the standard basis of $\mathbf{R}^3$, for projection onto the plane spanned by $\mathbf{v}_1$ and $\mathbf{v}_2$.

(iii) The matrix, with respect to the basis $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ of $\mathbf{R}^3$, for reflection in the plane spanned by $\mathbf{v}_1$ and $\mathbf{v}_2$.

(iv) The matrix, with respect to the basis $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ of $\mathbf{R}^3$, for the rotation about the line spanned by $\mathbf{v}_1$ which takes $\mathbf{v}_2$ to $\mathbf{v}_3$.

5) Let $E \subset \mathbf{R}^2$ denote the image of the unit circle (i.e. the set of $\mathbf{y} \in \mathbf{R}^2$ with $\mathbf{y}^T\mathbf{y} = 1$) under the linear transformation given by $\begin{bmatrix} 1 & 3/2 \\ 0 & 1 \end{bmatrix}$.

(a) Show that E is the set of vectors $\mathbf{x} \in \mathbf{R}^2$ such that $\mathbf{x}^T \begin{bmatrix} 4 & -6 \\ -6 & 13 \end{bmatrix} \mathbf{x} = 4$.

(b) Find an orthonormal eigenbasis $\mathbf{v}$, $\mathbf{w}$ of $\mathbf{R}^2$ for $\begin{bmatrix} 4 & -6 \\ -6 & 13 \end{bmatrix}$.

(c) Show that there are two real numbers l, m, which you should identify explicitly, such that E is the set of vectors $a\mathbf{v} + b\mathbf{w}$ with $l a^2 + mb^2 = 4$.

(d) What is the nearest that any point of E comes to $\mathbf{0}$?

6) Consider the continuous dynamical system

$$\left\{ \begin{aligned} \frac{dx}{dt} &= y \\ \frac{dy}{dt} &= (1 - x^2)y - x \end{aligned} \right\}.$$

(a) Sketch the nullclines for this linear system in the region $-4 \le x, y \le +4$, identify the equilibrium point and indicate the approximate direction of the trajectories in each region of your diagram. DO NOT mark any trajectories.

(b) Linearize these equations near the equilibrium solution. Is this a stable or unstable equilibrium? Describe briefly and qualitatively the behavior of the trajectories near this equilibrium solution.

(c) Now, sketch a possible phase portrait (system of trajectories) for the original dynamical system in the region $-4 \le x, y \le +4$.

7) Consider the linear transformation $T : C^\infty \to C^\infty$ given by $T(f) = \dfrac{df}{dt} - f$.

(a) Show that $e^t \int_0^t e^{-s} g(s)\, ds$ is a solution to the differential equation $\dfrac{df}{dt} - f = g$.

(b) What is the image and what is the kernel of T?

(c) What is the image and what is the kernel of T^2?

(d) Find all solutions of the equation

$$\frac{d^2 f}{dt^2} - 2\frac{df}{dt} + f = e^{-t}.$$

8) (a) Find the Fourier series for the function $f : [-p, p] \to \mathbf{R}$ given by $f(t) = t$.

(b) Solve the heat equation $\dfrac{\partial T(x,t)}{\partial t} = \dfrac{\partial^2 T(x,t)}{\partial x^2}$ in the region $t \ge 0$, $0 \le x \le p$ and subject to the initial conditions

$$T(x, 0) = x \quad \text{for } 0 < x < p$$

$$T(0, t) = T(p, t) = 0 \quad \text{for } t > 0.$$