Math 21b, 1997-98, 1st Midterm

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Each question is worth 10 points. Exam is out of 50.

1 True or False (no explanation is necessary)

(a) T F : For any matrix A, image$(A) = $ image(rref(A)).

(b) T F : For any matrix A, dim(image$(A)) = $ rank(A).

(c) T F : If $\{\overrightarrow{v}_1, \overrightarrow{v}_2, \overrightarrow{v}_3\}$ are any linearly independent vectors in $\mathbb{R}^n$, then $\overrightarrow{v}_3$ is a linear combination of $\overrightarrow{v}_1$ and $\overrightarrow{v}_2$.

(d) T F : There is a 3×6 matrix whose kernel is two-dimensional.

(e) T F : There is a 2×2 matrix A such that $A^2 = -I_2$.

2 Each of the spaces V_i below is equal to one (and only one) of the spaces W_j. Find the matching space in each case.

$$V_1 = \text{image}\begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad W_1 = \text{image}\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$V_2 = \text{image}\begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix} \qquad W_2 = \text{image}\begin{bmatrix} 1 & 0 \\ -1 & -1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$V_3 = \ker\begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \qquad W_3 = \ker\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$$V_4 = \ker\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} \qquad W_4 = \ker\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{bmatrix}$$

$$V_5 = \text{span}\left(\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}\right) \qquad W_5 = \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right)$$

$$\begin{aligned} V_1 &= \\ V_2 &= \\ V_3 &= \\ V_4 &= \\ V_5 &= \end{aligned}$$

3 Let $A = \begin{bmatrix} 0 & 1 & 3 \\ -1 & 0 & 0 \\ 0 & 2 & 5 \end{bmatrix}$.

(a) Is A invertible? If so, find A^{-1}.

(b) Find A^2.

4 Let A be a 2×2 matrix ($A \neq I_2$) representing a shear parallel to a line L in the plane. Find:

(a) $\ker(A - I_2)$

(b) $\mathrm{image}(A - I_2)$

(c) $(A - I_2)^2$

5

(a) Let A be a 3×3 matrix for which

$$\mathrm{image}(A) = \mathrm{span}\left(\begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}\right)$$

What is $\mathrm{rank}(A)$? Give an example of such a matrix A.

(b) Let B be a 3×3 matrix for which $\ker(B) = \mathrm{span}\left(\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}\right)$.

What is $\mathrm{rank}(B)$? Give an example of such a matrix B.

(c) Could you have chosen A, B so that $\mathrm{rank}(AB) = 2$? Briefly justify your answer.