Math 21b, 1997-98, 1st Midterm
Solution Set

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Each question is worth 10 points. Exam is out of 50.

1 Questions True or False (no explanation is necessary)

(a) T F : For any matrix $A, \text{image}(A) = \text{image}(\text{rref}(A))$.

(b) T F : For any matrix $A, \dim(\text{image}(A)) = \text{rank}(A)$.

(c) T F : If $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ are any linearly independent vectors in $\mathbb{R}^n$, then $\vec{v}_3$ is a linear combination of $\vec{v}_1$ and $\vec{v}_2$.

(d) T F : There is a 3×6 matrix whose kernel is two-dimensional.

(e) T F : There is a 2×2 matrix A such that $A^2 = -I_2$.

1 Answers

(a) F Consider $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. Then $\text{rref}(A) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. so $\text{image}(A) = \text{span}(\vec{e}_2) \neq \text{span}(\vec{e}_1) = \text{image}(\text{rref}(A))$. In general, row operations preserve the kernel but change the image, while column operations preserve the image but change the kernel.

(b) T Fact 3.3.8

(c) F Consider

$$(\vec{v}_1, \vec{v}_2, \vec{v}_3) = \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$$

. You can't in general take any proper subset of a linearly dependent set and have it be linearly independent.

(d) F A 3×6 matrix is a linear transformation $\mathbb{R}^6 \to \mathbb{R}^3$, so at least three dimensions go away. Alternatively, if you consider them as a system of equations, you have at least three free variables, so that $\dim(\ker(A)) \geq 3$.

(e) T Take $A = iI_2$. Then $A^2 = i^2 I_2 = -I_2$. If you prefer to stay in real numbers, note that $-I_2$ is a rotation through 180°. So just let A be a rotation through 90°, so

$$A = \begin{bmatrix} \cos 90^\circ & -\sin 90^\circ \\ \sin 90^\circ & \cos 90^\circ \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Note that $(-A)^2 = A^2$, so $-A$ should work as well - what does $-A$ indicate in this case?

2 Question

Each of the spaces V_i below is equal to one (and only one) of the spaces W_j. Find the matching space in each case.

$$V_1 = \text{image}\begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad W_1 = \text{image}\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$V_2 = \text{image}\begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix} \qquad W_2 = \text{image}\begin{bmatrix} 1 & 0 \\ -1 & -1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$V_3 = \ker\begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \qquad W_3 = \ker\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$$V_4 = \ker\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} \qquad W_4 = \ker\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{bmatrix}$$

$$V_5 = \text{span}\left(\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}\right) \qquad W_5 = \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right)$$

2 Answers

$$\begin{aligned} V_1 &= W_4 \\ V_2 &= W_5 \\ V_3 &= W_1 \\ V_4 &= W_2 \\ V_5 &= W_3 \end{aligned}$$

There are many ways of doing this problem. One way is to note that put everything into spans, that is, find a basis for all of these subspaces, as discussed in Section 3.3 of the text. For V_1, V_2, W_1, W_2 this is straight-forward; just take the column vectors. For V_3, V_4, W_3, W_4, just follow the discussion in Section 3.3 of the text. Now compare bases, and you get your answers.

An alternative way is to notice that these are all 2 dimensional subspaces of $\mathbb{R}^4$. Thus, the image of one matrix will be the kernel of another if and only if their composition is zero. Converting V_5, W_5 to images of matrices (the opposite of above) and just multiplying some out will get you the answers.

3 Question

Let $A = \begin{bmatrix} 0 & 1 & 3 \\ -1 & 0 & 0 \\ 0 & 2 & 5 \end{bmatrix}$.

(a) Is A invertible? If so, find A^{-1}.

(b) Find A^2.

3 Answer

These are straight-forward calculations.

(a)

$$A^{-1} = \begin{bmatrix} 0 & -1 & 0 \\ -5 & 0 & 3 \\ 2 & 0 & -1 \end{bmatrix}$$

(b)

$$A^2 = \begin{bmatrix} -1 & 6 & 15 \\ 0 & -1 & -3 \\ -2 & 10 & 25 \end{bmatrix}$$

4 Question

Let A be a 2×2 matrix ($A \neq I_2$) representing a shear parallel to a line L in the plane. Find:

(a) $\ker(A - I_2)$

(b) $\text{image}(A - I_2)$

(c) $(A - I_2)^2$

4 Answer

(a) L

We want to find all vectors such that $(A - I_2)\overrightarrow{v} = 0$. By linearity, we get $(A\overrightarrow{v} - I_2\overrightarrow{v} = 0, A\overrightarrow{v} = I_2\overrightarrow{v} = \overrightarrow{v}$, so we want to find all vectors such that $A\overrightarrow{v} = \overrightarrow{v}$, in other words, everything that doesn't get changed by applying the matrix.

When you have a shear, everything that is not on the line of the shear gets shifted by some amount, and the only things that don't get changed are the vectors on the line, namely the vectors in L.

(b) L

This follows from the definition of shear (definition 2.2.4).

(c) 0

$\text{image}(A - I_2) = \ker(A - I_2)$, so every vector gets sent to the kernel by A, so when you apply A again, it goes to zero.

5 Question

(a) Let A be a 3×3 matrix for which

$$\text{image}(A) = \text{span}\left(\begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} \right)$$

What is rank(A)? Give an example of such a matrix A.

(b) Let B be a 3×3 matrix for which $\ker(B) = \text{span}\left(\begin{bmatrix}1\\0\\2\end{bmatrix}, \begin{bmatrix}0\\1\\-1\end{bmatrix}\right)$.

What is rank(B)? Give an example of such a matrix B.

(c) Could you have chosen A, B so that $\text{rank}(AB) = 2$? Briefly justify your answer.

5 Answer

(a) By Fact 3.3.8, $\text{rank}(A) = \dim(\text{image}(A))$, so

$$\text{rank}(A) = \dim(\text{image}(A)) = \dim\left(\text{span}\left(\begin{bmatrix}3\\2\\0\end{bmatrix}, \begin{bmatrix}-1\\0\\2\end{bmatrix}\right)\right)$$

Since these 2 vectors are linearly independent, the dimension of their span is 2, so $\text{rank}(A) = 2$.

The image of a matrix is the span of its column vectors, so just let the columns of A be $\begin{bmatrix}3\\2\\0\end{bmatrix}, \begin{bmatrix}-1\\0\\2\end{bmatrix}$ and some linear combination, so that you get a 3×3 matrix. An example is

$$A = \begin{bmatrix}3 & 2 & 0\\-1 & 0 & 2\\0 & 0 & 0\end{bmatrix}$$

(b) By Fact 3.3.5, $\dim(\ker(B)) + \dim(\text{image}(B)) = 3$. Again, these vectors are linearly independent, so $\dim(\ker(B)) = 2$, so $3 - 2 = 1 = \dim(\text{image}(B)) = \text{rank}(B)$.

The rows of B are of the form $\begin{bmatrix}a & b & c\end{bmatrix}$, and must satisfy

$$\begin{bmatrix}a & b & c\end{bmatrix}\begin{bmatrix}1\\0\\2\end{bmatrix} = \begin{bmatrix}a & b & c\end{bmatrix}\begin{bmatrix}0\\1\\-1\end{bmatrix} = 0$$

This resolves into the system of linear equations:

$$\left|\begin{array}{ccccccc}a & & & + & 2c & = & 0\\ & & b & - & c & = & 0\end{array}\right| = \left|\begin{array}{ccc}a & = & -2c\\ b & = & c\end{array}\right|$$

Since rank$B$ = 1, we only need one row, so we can plug in $c = 1$, then fill out with zeros and get:

$$B = \begin{bmatrix} -2 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(c) Nope. The easiest way to see this is that B has already killed two dimensions, so the greatest possible dimension afterwards is 1. Alternatively, note that $\ker(AB) \supseteq \ker(B)$, so $\dim(\ker(AB)) \geq \dim(\ker(B))$, so $\dim(\ker(AB)) \geq 2$, and by Fact 3.3.5 & 3.3.8,

$$\text{rank}(AB) = \dim(\text{image}(AB)) = 3 - \dim(\ker(AB)) \leq 1$$