Last Name: _____

First Name: _____

Mathematics 21b

Second Exam
November 17, 1997

Your Section (circle one):

Otto	Richard	Seth
Bretscher	Taylor	Padowitz
MWF 10	MWF 11	TuTh 10

Question	Score
1	
2	
3	
4	
5	
Total	

The exam is worth 50 points.
No calculators are allowed.
Justify your answers carefully (except in Questions 1 and 2).
For Questions 3–5, no credit can be given for unsubstantiated answers.
Write your final answers in the spaces provided.

1. (12 points)
True or False (no explanation is necessary).

T F : If A is a square matrix such that A^{10} is the zero matrix, then 0 is an eigenvalue of A.

T F : For any square matrix A, the transpose of $A^T A$ is AA^T.

T F : If a 2×2 matrix A is not invertible, then $\mathrm{Tr}(A)$ is an eigenvalue of A.

T F : If A is an $m \times n$ matrix with $AA^T = I_m$, then the rows of A are orthogonal unit vectors.

T F : The 2×2 matrix P representing the orthogonal projection onto a line L in the plane is an orthogonal matrix.

T F : $\det\begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} = -1$.

2. (8 points)

Consider the discrete dynamical system

$$x(t+1) = \begin{bmatrix} 1 & 1 \\ 1 & k \end{bmatrix} x(t),$$

where $k > 0$ is an arbitrary positive real number. For each phase portrait below, find a possible value of k such that the phase portrait of the system is of the pictured type, or state that no such k exists.

(a)

$k =$

(b)

$k =$

(c)

$k =$

(d)

$k =$

3. (10 points)

Consider the linear system $A\vec{x} = \vec{b}$ with

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix} \quad \text{and} \quad \vec{b} = \begin{bmatrix} 3 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

(a) Find all least-squares solutions $\vec{x}$ to this linear system.

(b) Compute $\|A\vec{x} - \vec{b}\|$ for these solutions.

4. (10 points) Let $A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 3 \end{bmatrix}$.

(a) Find the eigenvalues of A together with their algebraic multiplicities.

(b) Calculate the geometric multiplicity of each eigenvalue.
Is there an eigenbasis for A?

5. (10 points)

Let A be a 3×5 matrix and B be a 5×3 matrix such that $AB = I_3$. In this exercise we consider the eigenvalues and eigenvectors of the matrix BA.

Note: Parts (a), (b), and (c) are logically independent.

(a) Explain why $\text{rank}(A) = 3$ and $\text{rank}(B) = 3$.

(b) Show that the nonzero vectors in $\text{Ker}(A)$ are eigenvectors of BA. What is the corresponding eigenvalue?

(c) Show that the nonzero vectors in $\mathrm{Im}(B)$ are eigenvectors of BA.
What is the corresponding eigenvalue?

(d) Using your answers to parts (a), (b), and (c), list all the eigenvalues of BA together with their geometric multiplicities.
Is there an eigenbasis for BA?