① Solve for $T(x,t)$, $0 \le x \le \pi$, $t \ge 0$

$$\frac{\partial T}{\partial t} = \mu \frac{\partial^2 T}{\partial x^2} \qquad T(0,t) = T(\pi,t) = 0$$
$$T(x,0) = 4\sin x$$

We have

$$T(x,t) = \sum_{n=1}^{\infty} c_n e^{-n^2\mu t} \sin nx$$
$$T(x,0) = \sum_{n=1}^{\infty} c_n \sin nx$$

To find the coefficients c_n, we need to find the Fourier sine series of $T(x,0)$.

First, define the function over the interval $[-\pi, 0]$ so that it is an odd function:

$$\Theta(x) = 4\sin x, \quad -\pi \le x \le \pi$$

This Fourier series is easy to compute, since $\Theta(x)$ is already expressed as a sum of sines:

$$\Theta(x) = \sum_{n=1}^{\infty} c_n \sin nx, \quad c_1 = 4,\ c_n = 0 \text{ for } n > 1$$
$$= 4\sin x.$$

So $$T(x,t) = \sum_{n=1}^{\infty} c_n e^{-n^2\mu t} \sin nx = 4e^{-\mu t}\sin x.$$

② Solve for $T(x,t)$ $\quad 0 \le x \le \pi,\ t \ge 0$

$$\frac{\partial T}{\partial t} = \mu \frac{\partial^2 T}{\partial x^2} \qquad T(0,t) = T(\pi,t) = 0$$

$$T(x,0) = \begin{cases} 0 & x \le \pi/4 \\ 1 & \pi/4 < x < 3\pi/4 \\ 0 & x \ge 3\pi/4 \end{cases}$$

Again, $T(x,t) = \sum_{n=1}^{\infty} c_n e^{-n^2 \mu t} \sin nx$

To compute c_n, look at $T(x,0)$:

$$T(x,0) = \sum_{n=1}^{\infty} c_n \sin nx$$

Define $\Theta(x)$ to be equal to $T(x,0)$ over $[0,\pi]$ and an odd function over $[-\pi,\pi]$

$$\Theta(x) = \begin{cases} 1 & \pi/4 < x < 3\pi/4 \\ -1 & -3\pi/4 < x < -\pi/4 \\ 0 & \text{otherwise},\ -\pi \le x \le \pi \end{cases}$$

Find the Fourier series of $\Theta(x)$: $\quad \Theta(x) = \sum_{n=1}^{\infty} c_n \sin nx$

$$c_n = \langle \Theta(x), \sin nx \rangle$$
$$= \frac{1}{\pi} \int_{-\pi}^{\pi} \Theta(x) \sin nx \, dx$$
$$= \frac{2}{\pi} \int_0^{\pi} \Theta(x) \sin nx \, dx = \frac{2}{\pi} \int_{\pi/4}^{3\pi/4} \sin nx \, dx$$
$$= \frac{2}{\pi} \left[-\frac{\cos nx}{n} \right]_{\pi/4}^{3\pi/4} = \frac{2}{n\pi} \left(-\cos \frac{3\pi}{4} n + \cos \frac{\pi}{4} n \right)$$

If n leaves a remainder of 1 or 7 upon division by 8,

$$c_n = +\frac{2\sqrt{2}}{n\pi}$$

If n leaves a remainder of 3 or 5 upon division by 8,

$$c_n = -\frac{2\sqrt{2}}{n\pi}$$

If n is even, $c_n = 0$

(These values are obtained by looking at $\cos \frac{\pi}{4} n$, $n = 1, 2, 3, \ldots$

$\sqrt{2}/2,\ 0,\ -\sqrt{2}/2,\ -1,\ -\sqrt{2}/2,\ 0,\ \sqrt{2}/2,\ 1,$ repeats, ...)

Using these values of c_n, we have

$$T(x,t) = \sum_{n=1}^{\infty} c_n e^{-n^2 \mu t} \sin nx$$

(3) If $T(x,t) = \frac{100}{\pi} x$, then:

$$\frac{\partial T}{\partial t} = 0 \quad (\text{since } T(x,t) \text{ is constant in } t)$$

$$\frac{\partial^2 T}{\partial x^2} = 0 \quad (\text{since } T(x,t) \text{ is linear in } x)$$

$$T(0,t) = \frac{100}{\pi} \cdot 0 = 0$$

$$T(\pi,t) = \frac{100}{\pi} \cdot \pi = 100$$

(4) Let $h(x,t)$ be a solution to $\frac{\partial T}{\partial t} = \mu \frac{\partial^2 T}{\partial x^2}$ subject to

$$T(0,t) = 0 \quad T(\pi,t) = 100 \quad T(x,0) = \frac{100}{\pi} x$$

Let $s(x,t)$ be a solution to $\frac{\partial T}{\partial t} = \mu \frac{\partial^2 T}{\partial x^2}$ subject to

$$T(0,t) = 0 \quad T(\pi,t) = 0 \quad T(x,0) = -\frac{100}{\pi} x$$

Then $h(x,t) + s(x,t)$ is a solution to $\frac{\partial T}{\partial t} = \mu \frac{\partial^2 T}{\partial x^2}$ subject to

$$T(0,t) = 0 \quad T(\pi,t) = 100 \quad T(x,0) = 0$$

By problem #3, we can use $h(x,t) = \frac{100}{\pi} x$.

To find $s(x,t)$, we can use the methods of this section:

Let $\Theta(x) = -\frac{100}{\pi} x$; note that $\Theta(x)$ is an odd function over $[-\pi,\pi]$, and it coincides with $s(x,0)$ over $[0,\pi]$

$$\Theta(x) = -\frac{200}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx$$ (see pg. 10)

so $$s(x,t) = -\frac{200}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} e^{-n^2 \mu t} \sin nx$$

and $$s(x,t) + h(x,t) = \frac{100}{\pi} x + -\frac{200}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} e^{-n^2 \mu t} \sin nx$$

As $t \to \infty$, $e^{-n^2 \mu t} \sin nx \to 0$, so

$$T(x,t) \to \frac{100}{\pi} x.$$

(5) if $T(x,t) = e^{-n^2\mu t} \cos nx$

then $\dfrac{\partial T}{\partial t} = -n^2 \mu e^{-n^2\mu t} \cos nx$

$$\frac{\partial^2 T}{\partial x^2} = -n^2 e^{-n^2 \mu t} \cos nx \text{ , so } \frac{\partial T}{\partial t} = \mu \frac{\partial^2 T}{\partial x^2}$$

$$\frac{\partial T}{\partial x}(0,t) = -ne^{-n^2\mu t} \sin nx \Big|_{0=x} = 0$$

$$\frac{\partial T}{\partial x}(\pi,t) = -ne^{-n^2\mu t} \sin nx \Big|_{\pi=x} = 0$$

(6) From #5, and by the linearity of the differential equation and boundary conditions, we can assume

$$T(x,t) = \sum_{n=0}^{\infty} c_n e^{-n^2\mu t} \cos nx$$

$$= c_0 e^{-0^2\mu t} + \sum_{n=1}^{\infty} c_n e^{-n^2\mu t} \cos nx = c_0 + \sum_{n=1}^{\infty} c_n e^{-n^2\mu t} \cos nx$$

so $T(x,0) = c_0 + \sum_{n=1}^{\infty} c_n \cos nx = x$

We need to find a Fourier cosine series for $T(x,0)$ (a Fourier series with only cosine and constant terms)

Define $\Theta(x)$ to be equal to $T(x,0)$ on $[0,\pi]$ and an <u>even</u> function on $[-\pi,\pi]$: $\Theta(x) = x$ if $0 \le x \le \pi$, $\Theta(x) = -x$ if $0 > x \ge -\pi$

So $\Theta(x) = |x|$. From the HW on Chapter 10.1,

$$|x| = \frac{\pi}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} + \sum_{n=1}^{\infty} c_n \cos nx \text{ , } c_n = \frac{-4}{\pi n^2} \text{ if } n \text{ odd, } 0 \text{ if } n \text{ even}$$

so $T(x,t) = \sum_{n=0}^{\infty} c_n e^{-n^2\mu t} \cos nx$, $c_0 = \frac{\pi}{2}$, $c_n = \frac{-4}{\pi n^2}$ if n odd, $c_n = 0$ if n even, $n > 0$

As $t \to \infty$, $c_n e^{-n^2 \mu t} \cos nx \to 0$ for all $n > 0$,
since $|c_n \cos nx| \leq |c_n| < 2$ while $e^{-n^2 \mu t} \to 0$

So as $t \to \infty$, $T(x,t) \to c_0 = \pi/2$.