Homework 3.1 : 10, 22, 34, 38, 44, 48

10. $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $\quad \mathrm{rref}(A) = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

So its kernel consists of solutions to $\left| \begin{matrix} x_1 & & -x_3 & = 0 \\ & x_2 & +2x_3 & = 0 \\ & & x_4 & = 0 \end{matrix} \right|$

The solutions are $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} t \\ -2t \\ t \\ 0 \end{bmatrix}$

Therefore, $\ker(A) = \mathrm{span}\left(\begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} \right)$

22. $A = \begin{bmatrix} 2 & 1 & 3 \\ 3 & 4 & 2 \\ 6 & 5 & 7 \end{bmatrix}$ $\quad \mathrm{rref}(A) = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$

Since rref(A) has two leading 1's, the image of A must be two dimensional. So the image of this transformation is a plane in $\mathbb{R}^3$.

34. The matrix $\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$ has a kernel spanned by $\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$ in $\mathbb{R}^3$.

38 a. Say $\vec{x} \in \ker A$. That is, $A\vec{x} = 0$. Then $A(A\vec{x}) = A(0) = 0$ so $\vec{x} \in \ker A^2$. What about the other direction? Consider $A = \begin{bmatrix} 1 & - \\ 1 & - \end{bmatrix}$. $A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ so every vector is in the kernel of A^2. But not every vector is in the kernel (for example, $A \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$). Therefore $\ker A \subseteq \ker A^2$, but it is not necessarily true that $\ker A^2 \subseteq \ker A$.

This argument can be extended to show that $\ker A^k \subseteq \ker A^{k+1}$ for all values of k.

b. Say $\vec{y} \in \mathrm{im}(A^2)$. Then $A^2\vec{x} = \vec{y}$ for some $\vec{x}$. Then $\vec{y} \in \mathrm{im}(A)$ since $A(A\vec{x}) = \vec{y}$. As with the A from (a), The only vector in $\mathrm{im}(A^2)$ is $\vec{0}$, but there are others in $\mathrm{im}(A)$. Therefore $\mathrm{im}(A^2) \subseteq \mathrm{im}(A)$. In fact, $\mathrm{im}(A^{k+1}) \subseteq \mathrm{im}(A^k)$ for all k.

44a. A a matrix, $B = \text{rref}(A)$.
$\ker A = \ker B$ because the way we determine the kernel of a matrix is by determining the rref form of it.

b. $\text{Im } A \neq \text{Im } B$. Consider $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Then $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.
$A \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, but the image of B consists only of vectors of the form $\begin{bmatrix} k \\ 0 \end{bmatrix}$.

48a. $A^2 = A$.
$A(Aw) = A^2 w = A(w)$
Since $w \in \text{im } A$, $A\vec{x} = w$ for some $\vec{x}$.
$w = A\vec{x} = A^2\vec{x} = A(A\vec{x}) = Aw$ so $w = Aw$

b. If $\text{rank}(A) = 0$, then A must be the zero matrix since it can be row reduced to $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and the only thing that can be row reduced to $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ itself.
If $\text{rank}(A) = 2$, then A can be row reduced to the identity matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. This means that all vectors in $\mathbb{R}^2$ are in the image of A, so $w = Aw$ for all $w \in \mathbb{R}^2$. Therefore A must equal the identity matrix.

c. If $\text{rank}(A) = 1$, then its image is spanned by one vector (call it $\vec{v_1}$, and its kernel is also spanned by one vector (call this $\vec{v_2}$). Note that these vectors must be distinct, and not scalar multiples of each other. So by 2.2.33, we can write $\vec{v} = k_1\vec{v_1} + k_2\vec{v_2}$ for any $\vec{v}$.
$$A(\vec{v}) = A(k_1\vec{v_1} + k_2\vec{v_2}) = k_1 A(\vec{v_1}) + k_2 A(\vec{v_2}) = k_1 A(\vec{v_1}) + 0 = k_1\vec{v_1} \ldots$$
Therefore, by 2.2.33, $T(x) = Ax$ is the projection onto $\text{im}(A)$ along $\ker(A)$