Homework 3.2 : 6, 18, 28, 38, 48, 52

6a. Is $V \cap W$ a subspace?

a. $\vec{0} \in V$ and $\vec{0} \in W$ (since V and W are subspaces) so $\vec{0} \in V \cap W$.

b. Say $\vec{x}, \vec{y} \in V \cap W$. Then $\vec{x}, \vec{y} \in V$ and $\vec{x}, \vec{y} \in W$, so $\vec{x} + \vec{y} \in V$ and $\vec{x} + \vec{y} \in W$ (since V and W are subspaces). Therefore $\vec{x} + \vec{y} \in V \cap W$

c. Say $\vec{x} \in V \cap W$ and k is an arbitrary scalar. $\vec{x} \in V$ and $\vec{x} \in W$ so $k\vec{x} \in V$ and $k\vec{x} \in W$ (since V and W are subspaces) $\Rightarrow k\vec{x} \in V \cap W$

Therefore $V \cap W$ is a subspace.

b. Let V be the x axis in $\mathbb{R}^2$ and W be the y axis in $\mathbb{R}^2$. These are subspaces of $\mathbb{R}^2$. $\begin{bmatrix}1\\0\end{bmatrix} \in V$ and $\begin{bmatrix}0\\1\end{bmatrix} \in W$, but $\begin{bmatrix}1\\0\end{bmatrix} + \begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix} \notin V \cup W$.

Therefore $V \cup W$ is not a subspace.

18. Notice that
$$2\begin{bmatrix}1\\1\\1\\1\end{bmatrix} - 3\begin{bmatrix}1\\2\\3\\4\end{bmatrix} + \begin{bmatrix}1\\4\\7\\10\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$
Therefore the vectors are linearly dependent.

28.
$$\begin{bmatrix}1&1&1\\1&2&5\\1&3&7\end{bmatrix} \xrightarrow[\text{(rref)}]{} \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$
Since the rref form has leading 1's in the 1^{st}, 2^{nd}, and 3^{rd} columns, the image is spanned by the 1^{st}, 2^{nd}, & 3^{rd} columns of the original matrix.

$$\text{image} = \text{span}\left(\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}1\\5\\7\end{bmatrix}\right)$$

38a. Let $\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n$ be the standard basis vectors for $\mathbb{R}^n$. Choose any $\vec{v} \in \mathbb{R}^n$ where $\vec{v} = \begin{bmatrix}v_1\\v_2\\\vdots\\v_n\end{bmatrix}$. Then $\vec{v} = v_1\vec{e}_1 + v_2\vec{e}_2 + \dots + v_n\vec{e}_n$

so the vectors $\vec{e}_1, \dots, \vec{e}_n, \vec{v}$ are linearly dependent regardless of what $\vec{v}$ is. Conclusion: there are at most n linearly independent vectors in $\mathbb{R}^n$.

b. Let's assume that $\vec{v}_1, \ldots, \vec{v}_m$ are linearly independent in V. If they don't span V, then there exists some vector $\vec{v} \in V$ that can't be written as a linear combination of them. But, by definition, this means that $\vec{v}_1, \ldots, \vec{v}_m, \vec{v}$ are m+1 linearly independent vectors in V which contradicts the fact that m is the largest number of linearly independent vectors that can be found in V.

c. Consider the matrix $\begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_m \\ | & | & & | \end{bmatrix}$ whose columns are the vectors $\vec{v}_1, \ldots, \vec{v}_m$.

The image of this matrix is the set of linear combinations of $\vec{v}_1, \ldots, \vec{v}_m$, so the image is exactly the subspace V.

48. Let $B = \begin{bmatrix} 3 & 4 & 5 \end{bmatrix}$ and $A = \begin{bmatrix} 4 & -3 & 0 \\ -5 & 0 & 3 \\ 0 & 5 & -4 \end{bmatrix}$

Then the kernel of B and the image of A are both equal to the plane $3x_1 + 4x_2 + 5x_3 = 0$.

52. Recall from #38 that we could make an $n \times m$ matrix whose image was the subspace V. To make this an $n \times n$ matrix, just add on $n-m$ more columns of 0's:

$$\begin{bmatrix} | & | & & | & 0 & 0 & & 0 \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_m & 0 & 0 & \cdots & 0 \\ | & | & & | & 0 & 0 & & 0 \end{bmatrix}$$
$$\underbrace{\qquad\qquad}_{n-m \text{ cols}}$$

This is an $n \times n$ matrix with image V. Therefore, the answer is TRUE.