Math 21b Section 4.3 Solution Set

Puneet Sethi

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Question 2

We need to show that

$$\vec{v} \cdot \vec{w} = \mathrm{L}(\vec{v}) \cdot \mathrm{L}(\vec{w})$$

But from Fact 4.3.6 we know that $\vec{v} \cdot \vec{w} = \vec{v}^T \cdot \vec{w}$ and from Fact 4.3.8 we know that $\mathrm{L}^T(\vec{v}) = \mathrm{L}^{-1}(\vec{v})$ if L is orthogonal. Equivalently, we know that $\mathrm{A}^T(\vec{v}) = \mathrm{A}^{-1}(\vec{v})$ for any orthogonal matrix A.

So we have

$$\begin{aligned} \mathrm{A}(\vec{v}) \cdot \mathrm{A}(\vec{w}) &= (\mathrm{A}(\vec{v}))^T(\mathrm{A}(\vec{w})) \\ &= (\vec{v})^T A^T A(\vec{w})) \\ &= (\vec{v})^T (A^{-1}A)(\vec{w})) \\ &= (\vec{v})^T (I_n)(\vec{w})) \\ &= (\vec{v})^T (I_n \vec{w}) \\ &= \vec{v} \cdot \vec{w} \end{aligned}$$

which proves what we wanted.

Question 6

Answer: YES

From Fact 4.3.4, we know that A is orthogonal $\Rightarrow$ A^{-1} is orthogonal also. But from Fact 4.3.8 we know that $A^T = A^{-1}$ so A^T is orthogonal as well.

Question 8

Part A

Answer: NO

Consider the matrix

$$A = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ with } A^T = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

This provides us with a counterexample.

We have

$$AA^T = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \text{ and } A^T A = \begin{bmatrix} 1 \end{bmatrix}$$

Part B

Answer: YES

By fact 4.3.8, we know that $AA^T = I_N \Rightarrow A$is orthogonal Then by the answer to Problem 6, we see that that the anwer is yes, since $A^T = A^{-1}$.

Question 18

Part A

Any matrix with 0 on the diagonal entries and, in general, with entries such that $a_{ij} = -a_{ji}$ will do.

For example, try

$$A = \begin{bmatrix} 0 & -1 & -1 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix}$$

Part B

If $A^T = -A$, then $A^2 = (-A)^2 = (A^T)^2 = (A^2)^T$, so we see that A^2 is symmetric.

Question 20

We find an orthonormal basis for W, and then use Fact 4.3.10 to find the matrix M.

Using Gram Schmidt, we find that $\vec{w}_1 = \frac{\vec{v}_1}{||\vec{v}_1||} = (\frac{1}{2})\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$ and that $\vec{w}_2 = \frac{\text{Proj}_{V_1}(\vec{v}2)}{||\text{Proj}_{V_1}(\vec{v}2)||} = (\frac{1}{10})\begin{bmatrix}-1\\7\\-7\\1\end{bmatrix}$

Now by Fact 4.3.10, we know that the matrix of the orthogonal projection onto W is AA^T, where $A = \begin{bmatrix}| & | \\ \vec{w}_1 & \vec{w}_2 \\ | & |\end{bmatrix}$

Mutiplying out, we get $AA^T = (\frac{1}{100})\begin{bmatrix}26 & 18 & 32 & 24\\18 & 74 & -24 & 32\\32 & -24 & 74 & 18\\24 & 32 & 18 & 26\end{bmatrix}$

Question 34

Part C

We need to show that if $M$and $N \in H$, then the product MN is in H as well.

If M is in H, then

$$M = \begin{bmatrix}A & -B^T\\B & A^T\end{bmatrix}$$

If N is in H, then

$$N = \begin{bmatrix}C & -D^T\\D & C^T\end{bmatrix}$$

Then we have

$$MN = \begin{bmatrix}AC + -B^TD & -AD^T + -B^TC^T\\BC + A^TD & -BD^T + A^TC^T\end{bmatrix}$$

Now, A, B, C, and D are all rotation-dilation matrices, and *thus all commute with each other.* Therefore, if we choose $E = AC - B^TD = CA - DB^T$ and $F = BC + A^TD = CB + DA^T$, we see that

$$MN = \begin{bmatrix}E & -F^T\\F & E^T\end{bmatrix}$$

and therefore is in H.

Part D

If M is in H, then $M = \begin{bmatrix} A & -B^T \\ B & A^T \end{bmatrix}$ and

$$M^T = \begin{bmatrix} A^T & B \\ -B^T & A \end{bmatrix} = \begin{bmatrix} C & -D^T \\ D & C^T \end{bmatrix}$$

for $C = A^T$ and $D = -B^T$, and thus M^T is also in H.

Part E

$$\begin{bmatrix} A & -B^T \\ B & A^T \end{bmatrix} \cdot \begin{bmatrix} A^T & B \\ -B^T & A \end{bmatrix} = \begin{bmatrix} AA^T + (B^2)^T & AB - B^T A \\ BA^T - A^T B^T & B^2 + A^T A \end{bmatrix}$$

Part F

The columns of M are orthogonal, so that as long they are not equal to 0 they are linearly independent and M is invertible. So M is invertible as long as at least one of p,q,r,s are not equal to 0, which is true as long as either A or B is invertible.

If M is invertible, then from using the formula for the inverse of 2 by 2 matrix as guide, we see that

$$M^{-1} = \frac{1}{\det(AA^T + BB^T)} \begin{bmatrix} A^T & B \\ -B^T & A \end{bmatrix}$$

This works, as a check, note that

$$MM^{-1} = M^{-1}M = \frac{1}{p^2 + q^2 + r^2 + s^2} \begin{bmatrix} p^2 + q^2 + r^2 + s^2 & 0 \\ 0 & p^2 + q^2 + r^2 + s^2 \end{bmatrix} = I_2$$

Part G

It NOT true that $MN = MN$ for all M, N in H.

$$MN = \begin{bmatrix} AC + -B^T D & -AD^T + -B^T C^T \\ BC + A^T D & -BD^T + A^T C^T \end{bmatrix}$$

and

$$NM = \begin{bmatrix} CA + -D^T B & -CB^T + -D^T A^T \\ DA + C^T B & -DB^T + C^T A^T \end{bmatrix}$$

One way to prove this is to prove that in general, it is not true that $BC + A^T D = DA + C^T B$ For example, take $A = B = D = I_2$ and $C = \begin{bmatrix} x & -y \\ y & x \end{bmatrix}$

Then
$$BC + A^T D = C + D = \begin{bmatrix} x+1 & -y \\ y & x+1 \end{bmatrix}$$
and
$$DA + C^T B = D + C^T = \begin{bmatrix} x+1 & y \\ -y & x+1 \end{bmatrix}$$