Math 21b Solution Set Section 5.2

Puneet Sethi

March 17 2000

Question 3

Recall that adding a multiple of a row to another does not change the value of the determinant, by Fact 5.2.4. Using row reduction, we can then proceed to caluculate the determinant.

Adding multiples of the first row to the other rows gives us

$$\begin{bmatrix} 1 & -1 & 2 & -2 \\ 0 & 1 & 3 & 4 \\ 0 & 3 & 10 & 14 \\ 0 & 4 & 14 & 29 \end{bmatrix}$$

and adding multiples of the second row to the other rows yields:

$$\begin{bmatrix} 1 & 0 & 5 & 2 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 2 & 13 \end{bmatrix}$$

and adding a multiple of the third row to the fourth row yields:

$$\begin{bmatrix} 1 & 0 & 5 & 2 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 9 \end{bmatrix}$$

We now have an upper triangular matrix whose determinant is equal to the determinant of the original matrix. The determinant of this matrix is just the product of the diagonal entries, or 9.

Question 4

Using Laplace expansion, we look to expand by rows or columns with 0's in order to simplify our computations.

Expanding by the first column, we find that $\det(A) = (3)\det\left(\begin{bmatrix} 2 & 1 & 0 & 1 \\ 0 & 2 & 0 & 2 \\ 5 & 3 & 9 & 9 \\ 7 & 4 & 0 & 1 \end{bmatrix}\right) =$

$(3)(9)\det\left(\begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & 2 \\ 7 & 4 & 1 \end{bmatrix}\right) = $ (expanding along the first column) $(3)(9)(2(-7)) =$

-378

1 Question 6

We can use row reduction to figure out $\det(M_n)$ for small n, and then extend to arbitrary n using induction.

For $n = 1$, the determinant is clearly 1. For $n = 2$, $M_2 = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ and so $\det(M_2) = 1(2) - 1(1) = 1$.

For $n = 3$, we see that $M_3 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$ and by subtracting the first row from the second and third rows, we obtain a matrix N with the same determinant. Now, $N = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$, and by expanding along the first column we see that $\det(N) = \det(M_2) = 1$.

Similarly, for any n, we can subtract the first row of M_n from the other rows to obtain:

$$\det(M_n) = \det\left(\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 0 & & \vdots & \\ \vdots & \cdots & M_{n-1} & \cdots \\ 0 & & \vdots & \end{bmatrix}\right)$$

, and using Laplace expansion along the first column we see that

$$\det(M_n) = \det(M_{(n-1)}) = 1$$

Question 16

Part A

Expanding along the first column, we see that $f(t)$ is a polynomial in t whose degree is 2, so $f(t)$ is quadratic. Again using Laplacian expansion, we see that the coefficient of t^2 is $\det\left(\begin{bmatrix} 1 & 1 \\ a & b \end{bmatrix}\right) = (b - a)$ so that $f(t) = (b - a)t^2 - (b^2 - a^2)t - (ab)(b - a)$.

Part B

Substituting a or b for t, we find that the given matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ t & a & b \\ t^2 & a^2 & b^2 \end{bmatrix}$ has linearly dependent columns, so that for $t = a$ or $t = b$ we have $\det(A) = 0$.

Since f(t) is a quadratic equation, it has two roots which we've identified as a and b, so that f$(t) = k(t - a)(t - b)$ for some k. Since the coefficient of t^2 is k, from part A we observe that $k = (b - a)$.

Part C

From the factorization of f(t), we see that if $t \neq a$ and $t \neq b$, then f$(t) \neq 0$ and thus the matrix in invertible.

Question 36

Consider an element of a pattern a_{ij}. If there are k entries of the pattern to the left and above a_{ij}, there must be $(j - 1) - k$ entries to the left and below a_{ij}. Moreover, if there are k entries to the left and above a_{ij}, there must be $(i - 1) - k$ entries to the right and above a_{ij}. Thus, the number of inversions a_{ij} is involved in is:

$$((j - 1) - k)) + ((i - 1) - k) = (i + j) - 2(k + 1)$$

If (i+j) is even, then subtracting an even number from an even number will yield an even number.

If (i+j) is odd, then subtracting an even number from an odd number will yield an odd number.

Question 40

Using the hint, we note that since A and B are 2 by 2 matrices, then $(A+tB)$ has a t in each entry, and thus $\det(A+tB)$ is a quadratic equation and $(\det(A+tB))^2$ has degree 4 if we regard it as a polynomial in the variable t.

From Exercise 39, note that if an invertible A has integer entries, then $|\det(A)| = 1$ if and only if the entries of A^{-1} are integers.

Since A, $A + B$, ..., $A + 4B$ satisfy these criteria, we know that $|\det(A + tB)| = 1$ for $0 \leq t \leq 4$, so that f$(t) = 0$ for $0 \leq t \leq 4$. Now, since f(t) has degree 4 but has at least 5 zeroes, we know that f$(t) = 0$ for all t by elementary algebra, so that in particular f$(5) = 0$.