Homework 6.5: 8, 12, 22, 28, 40afg, 42

8: $A = \begin{bmatrix} 1 & -0.2 \\ 0.1 & 0.7 \end{bmatrix}$ $\quad f_A(\lambda) = \lambda^2 - (1+0.7)\lambda + (1\times 0.7 - 0.1\times -0.2)$

$= \lambda^2 - 1.7\lambda + 0.72 = (\lambda - 0.9)(\lambda - 0.8)$

Since the eigenvalues of A are 0.9 and 0.8, and $|0.9| < 1$ and $|0.8| < 1$, the zero state is a stable equilibrium for this dynamical system.

12. $A = \begin{bmatrix} 0.6 & k \\ -k & 0.6 \end{bmatrix}$ $\quad f_A(\lambda) = \lambda^2 - 1.2\lambda + 0.36 + k^2$

To find the eigenvalues, use the quadratic formula:

$$\lambda = \frac{1.2 \pm \sqrt{(1.44 - 1.44 - 4k^2)}}{2} = 0.6 \pm ki$$

We want $1 > |\lambda| = \sqrt{(0.6)^2 + k^2}$, so we need $k^2 < 1 - .6^2 = 0.64$

Therefore, $-0.8 < k < 0.8$ will make the zero state a stable equilibrium for A.

22. $A = \begin{bmatrix} 7 & -15 \\ 6 & -11 \end{bmatrix}$ $\quad \vec{x}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$f_A(\lambda) = \lambda^2 + 4\lambda + 13 \implies \lambda_{1,2} = -2 \pm 3i$. Say $\lambda_1 = -2 + 3i$.

$\ker(\lambda_1 I - A) = \ker \begin{bmatrix} -9+3i & 15 \\ -6 & 9+3i \end{bmatrix} = \ker \begin{bmatrix} 2 & -3-i \\ 0 & 0 \end{bmatrix} = \operatorname{span}\left(\begin{bmatrix} 3+i \\ 2 \end{bmatrix} \right)$

$\begin{bmatrix} 3+i \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} i = \vec{v} + i\vec{w}$

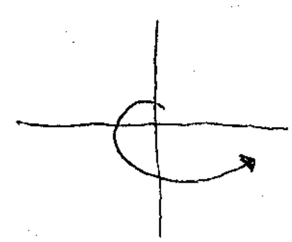
$x_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 3 \\ 2 \end{bmatrix} - \frac{3}{2}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\qquad r = |\lambda_1| = \sqrt{2^2 + 3^2} = \sqrt{13}$

What is ϕ? $-2 + 3i = \sqrt{13}(\cos\phi + i\sin\phi) \Rightarrow \phi \approx 123.7°$ or 2.16 rad.

So by Fact 6.5.3,

$$\vec{x}(t) = (\sqrt{13})^t \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \cos 123.7t & -\sin 123.7t \\ \sin 123.7t & \cos 123.7t \end{bmatrix} \begin{bmatrix} -\frac{3}{2} \\ \frac{1}{2} \end{bmatrix}$$

$$= (\sqrt{13})^t \begin{bmatrix} -5\sin 123.7t \\ -3\sin 123.7t + \cos 123.7t \end{bmatrix}$$



28. The zero state is a stable equilibrium of A. What about $A-2I_n$?

$$(A-2I_n)^k = A^k + \binom{k}{1}A^{k-1}(-2I)^1 + \binom{k}{2}A^{k-2}(-2I)^2 + \dots + \binom{k}{k-1}A^1(-2I)^{k-1} + (-2I)^k$$

For $A-2I_n$ to have a stable equilibrium in the zero state, $\lim_{k\to\infty}(A-2I_n)^k$ must go to $[0]$.

However, no matter how quickly $A^k \to [0]$, there will always be the nonzero $(-2I)^k$ added on at the end, so

$$\lim_{k\to\infty}(A-2I_n)^k \neq [0]$$

and $A-2I_n$ does not have a stable equilibrium in the zero state.

40a.

$$A = \begin{bmatrix} p & -q & -r & -s \\ q & p & s & -r \\ r & -s & p & q \\ s & r & -q & p \end{bmatrix} \rightarrow A^TA = \begin{bmatrix} p^2+q^2+r^2+s^2 & 0 & 0 & 0 \\ 0 & p^2+q^2+r^2+s^2 & 0 & 0 \\ 0 & 0 & p^2+q^2+r^2+s^2 & 0 \\ 0 & 0 & 0 & p^2+q^2+r^2+s^2 \end{bmatrix}$$

f. $2183 = 37 \times 59 = (1^2+2^2+4^2+4^2)(3^2+3^2+4^2+5^2)$

In the matrix A, let $p=3, q=3, r=4, s=5$. Then $A \cdot \begin{bmatrix}1\\2\\4\\4\end{bmatrix} = \begin{bmatrix}13\\13\\18\\39\end{bmatrix}$

This indicates that $2183 = 13^2+13^2+18^2+39^2$, and checking the arithmetic we see that this is, in fact, the case.

g. Any positive integer is the product of primes, and any prime is the sum of four squares. Let n be any positive integer, and write $n = p_1p_2\cdots p_k$ where $p_1,\dots,p_k$ are primes. Also, write

$$p_i = a_i^2+b_i^2+c_i^2+d_i^2 \text{ for each } 1\le i\le k.$$

If we let

$$A_i = \begin{bmatrix} a_i & -b_i & -c_i & -d_i \\ b_i & a_i & d_i & -c_i \\ c_i & -d_i & a_i & b_i \\ d_i & c_i & -b_i & a_i \end{bmatrix}$$

for each $1\le i\le k-1$, then the product

$$A_1A_2\cdots A_{k-1}\begin{bmatrix}a_k\\b_k\\c_k\\d_k\end{bmatrix}$$ will give a vector $\begin{bmatrix}n_1\\n_2\\n_3\\n_4\end{bmatrix}$ and $n=n_1^2+n_2^2+n_3^2+n_4^2$

42a. $\begin{bmatrix} 1 & -k \\ k & 1-k^2 \end{bmatrix} = A$

$\lambda^2 - (2-k^2)\lambda + 1 = f_A(\lambda)$.

b. We know $\lambda_1 \lambda_2 = 1$, and since k is small,

$$(2-k^2)^2 - 4 < 0$$

Therefore the eigenvalues are complex and $\lambda_1 = \overline{\lambda_2}$, so we have $\lambda_1 \lambda_2 = \overline{\lambda_2} \lambda_2 = |\lambda_2| = 1 = |\lambda_1|$

This means that the trajectories are ellipses, as claimed.