Homework 7.1 : 2, 4, 6, 12, 14, 24

2. To find a, b, c, d in $\begin{bmatrix}-4\\4\\3\\-3\end{bmatrix} = a\begin{bmatrix}1\\2\\3\\4\end{bmatrix} + b\begin{bmatrix}5\\6\\7\\8\end{bmatrix} + c\begin{bmatrix}2\\3\\5\\7\end{bmatrix} + d\begin{bmatrix}1\\9\\9\\4\end{bmatrix}$

we need to solve the system
$$\left|\begin{array}{l} a + 5b + 2c + d = -4 \\ 2a + 6b + 3c + 9d = 4 \\ 3a + 7b + 5c + 9d = 3 \\ 4a + 8b + 7c + 4d = -3 \end{array}\right|$$

$\Rightarrow a = 2, b = -1, c = -1, d = 1$

Then the coordinate vector of $\begin{bmatrix}-4\\4\\3\\-3\end{bmatrix}$ in this system is $\begin{bmatrix}2\\-1\\-1\\1\end{bmatrix}$

4. $A = \begin{bmatrix}7 & -1\\-6 & 8\end{bmatrix}$ $S = \begin{bmatrix}1 & -1\\2 & 3\end{bmatrix}$ $S^{-1} = \frac{1}{5}\begin{bmatrix}3 & 1\\-2 & 1\end{bmatrix} = \begin{bmatrix}0.6 & 0.2\\-0.4 & 0.2\end{bmatrix}$

$B = S^{-1}AS = \begin{bmatrix}5 & 0\\0 & 10\end{bmatrix}$

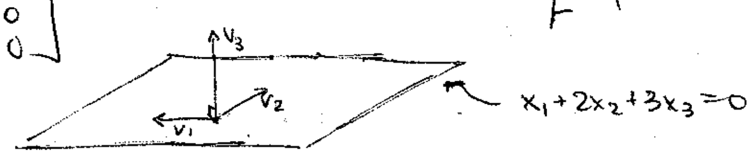
6a. $T: \mathbb{R}^3 \to \mathbb{R}^3$ is reflection in the plane $x_1 + 2x_2 + 3x_3 = 0$.

$\vec{v}_1 = \begin{bmatrix}1\\1\\-1\end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix}-1\\2\\-1\end{bmatrix}$ $\vec{v}_3 = \begin{bmatrix}1\\2\\3\end{bmatrix}$

Notice that $\vec{v}_1$ and $\vec{v}_2$ are both in the plane, while $\vec{v}_3$ is normal to it. So $T(\vec{v}_1) = \vec{v}_1$, $T(\vec{v}_2) = \vec{v}_2$, $T(\vec{v}_3) = \vec{0}$.

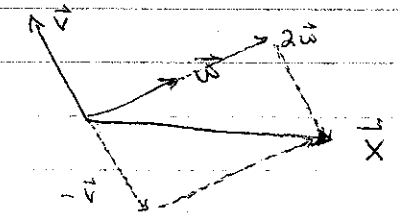
The matrix B of T relative to $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is $\begin{bmatrix} | & | & | \\ [T(v_1)]_\beta & [T(v_2)]_\beta & [T(v_3)]_\beta \\ | & | & | \end{bmatrix}$

so $B = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 0\end{bmatrix}$



b. $A = SBS^{-1} = \begin{bmatrix}1 & -1 & 1\\1 & 2 & 2\\-1 & -1 & 3\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 0\end{bmatrix}\begin{bmatrix}1 & -1 & 1\\1 & 2 & 3\\-1 & -1 & 3\end{bmatrix}^{-1} = \frac{1}{14}\begin{bmatrix}13 & -2 & -3\\-2 & 10 & -6\\-3 & -6 & 5\end{bmatrix}$

12. $[\vec{x}]_\beta = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

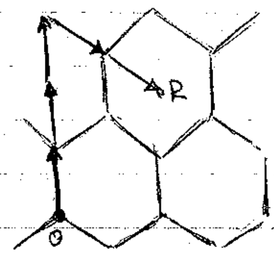


$\vec{x} = -\vec{v} + 2\vec{w}$

14a. $[\vec{OP}]_\beta \Rightarrow \vec{w} + 2\vec{v}$ so $[\vec{OP}]_\beta = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$[\vec{OQ}]_\beta \Rightarrow \vec{v} + 2\vec{w}$ so $[\vec{OQ}]_\beta = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

b.



R is at the center of a tile.

c. $[\vec{OS}]_\beta = \begin{bmatrix} 17 \\ 13 \end{bmatrix}$ If u is at a vertex, $u + 3\vec{v}$ will be, as will $u + 3\vec{w}$. Notice $17\vec{v} + 13\vec{w} = (2\vec{v} + \vec{w}) + 5(3\vec{v}) + 4(3\vec{w})$
Therefore S is at a vertex.

24a. $\vec{v}_0 + \vec{v}_1 + \vec{v}_2 + \vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} + \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \vec{0}$

b. $\vec{v}_0 = -\vec{v}_1 - \vec{v}_2 - \vec{v}_3$, so if $\beta = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$, $[\vec{v}_0]_\beta = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}$

c. $T(\vec{v}_0) = \vec{v}_3$, $T(\vec{v}_3) = \vec{v}_1$, $T(\vec{v}_1) = \vec{v}_0$.

$T(\vec{v}_2) = T(-\vec{v}_0 - \vec{v}_1 - \vec{v}_3) = -T(\vec{v}_0) - T(\vec{v}_1) - T(\vec{v}_3) = -\vec{v}_3 - \vec{v}_0 - \vec{v}_1 = \vec{v}_2$

So T is a rotation by $120°$ about the line spanned by $\vec{OP}_2$.

$$B = \begin{bmatrix} | & | & | \\ [T(\vec{v}_1)]_\beta & [T(\vec{v}_2)]_\beta & [T(\vec{v}_3)]_\beta \\ | & | & | \end{bmatrix} = \begin{bmatrix} | & | & | \\ [\vec{v}_0]_\beta & [\vec{v}_2]_\beta & [\vec{v}_1]_\beta \\ | & | & | \end{bmatrix} = \begin{bmatrix} -1 & 0 & 1 \\ -1 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

$f_B(\lambda) = \lambda^3 - 1$, so $\lambda_1 = 1$, $\lambda_{2,3} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i = \text{cis}(\pm 120°)$

$B^3 = I_3$ and this makes sense since B is rotation by $120°$ and if you do this three times, you should get back to the start