§7.3 pp. 409 – 413 # 2, 10, 12, 16, 26, 36 Bryant Mathews

Due Friday, April 14

(2) $(1-\lambda)^2 - 1 = 0 \iff \lambda^2 - 2\lambda = \lambda(\lambda - 2) = 0.$
$\iff \lambda = 0$ or $\lambda = 2$,

so the eigenvalues are 0 and 2,

with eigenvectors $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ respectively.

To get an orthonormal basis, we divide these vectors by their lengths, giving $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

(10) $\det(A - \lambda I) = \det \begin{pmatrix} 1-\lambda & -2 & 2 \\ -2 & 4-\lambda & -4 \\ 2 & -4 & 4-\lambda \end{pmatrix}$

$$= (1-\lambda)(4-\lambda)^2 + 16 + 16 - 16(1-\lambda) - 4(4-\lambda) - 4(4-\lambda)$$
$$= (16 - 24\lambda + 9\lambda^2 - \lambda^3) + 32 - 48 + 24\lambda$$
$$= -\lambda^3 + 9\lambda^2 = -\lambda^2(\lambda - 9),$$ so

A has eigenvalues 0, 0, 9.

The respective eigenvectors can be chosen to be

$\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$. We know these are orthogonal,

since A is symmetric.

We let $\Lambda = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 9 \end{pmatrix}$, $S = \begin{pmatrix} 2 & 0 & 1 \\ 1 & 1 & -2 \\ 0 & 1 & 2 \end{pmatrix}$.

(12) a) An orthonormal eigenbasis will be given by a unit vector in the direction of $\vec{v}$ and two unit vector that are perpendicular, and span the plane perpendicular to $\vec{v}$. We choose:

$$\vec{v}_1 = \frac{\vec{v}}{|\vec{v}|} = \frac{1}{\sqrt{5}}\begin{pmatrix}1\\0\\2\end{pmatrix}, \quad \vec{v}_2 = \frac{1}{\sqrt{5}}\begin{pmatrix}2\\0\\-1\end{pmatrix}, \quad \vec{v}_3 = \begin{pmatrix}0\\1\\0\end{pmatrix}.$$

b) $B = \begin{pmatrix}-1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$ eigenvalues should be $-1, 1, 1$.

c) Let $S = \begin{pmatrix}1/\sqrt{5} & 2/\sqrt{5} & 0\\ 0 & 0 & 1\\ 2/\sqrt{5} & -1/\sqrt{5} & 0\end{pmatrix}$.

Then $S^{-1} = S^t = \begin{pmatrix}1/\sqrt{5} & 0 & 2/\sqrt{5}\\ 2/\sqrt{5} & 0 & -1/\sqrt{5}\\ 0 & 1 & 0\end{pmatrix}$.

Thus $A = (SB)S^{-1} = \begin{pmatrix}-1/\sqrt{5} & 2/\sqrt{5} & 0\\ 0 & 0 & 1\\ -2/\sqrt{5} & -1/\sqrt{5} & 0\end{pmatrix}\begin{pmatrix}1/\sqrt{5} & 0 & 2/\sqrt{5}\\ 2/\sqrt{5} & 0 & -1/\sqrt{5}\\ 0 & 1 & 0\end{pmatrix}$

$$= \begin{pmatrix}3/5 & 0 & -4/5\\ 0 & 1 & 0\\ -4/5 & 0 & -3/5\end{pmatrix}.$$

(16) a) 0, with multiplicity 4:

$$\begin{pmatrix}1\\-1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\-1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\-1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\\-1\end{pmatrix}.$$

5, with multiplicity 1: $\begin{pmatrix}1\\1\\1\\1\\1\end{pmatrix}$.

b) 2, with multiplicity 4:

$$\begin{pmatrix}1\\-1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\-1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\-1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\\-1\end{pmatrix}.$$

7, with multiplicity 1: $\begin{pmatrix}1\\1\\1\\1\\1\end{pmatrix}$.

c) $$\det(B) = \det\begin{pmatrix}2 & & & & 0\\ & 2 & & & \\ & & 2 & & \\ & & & 2 & \\ 0 & & & & 7\end{pmatrix} = 2^4 \cdot 7 = 112.$$

(26) $$J_n = \begin{pmatrix}0 & & 1\\ & \iddots & \\ 1 & & 0\end{pmatrix}.$$

<u>n even</u>: $\lambda = 1$ with mult. $n/2$
$\lambda = -1$ with mult. $n/2$

<u>n odd</u>: $\lambda = 1$ with mult. $\frac{n+1}{2}$
$\lambda = -1$ with mult. $\frac{n-1}{2}$.

<u>eigenvectors</u>:

<u>$\lambda = 1$</u>: $\begin{pmatrix}1\\0\\\vdots\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\0\\\vdots\\0\\1\\0\end{pmatrix}, \dots, \begin{pmatrix}0\\\vdots\\0\\1\\1\\0\\\vdots\\0\end{pmatrix}$ or $\begin{pmatrix}0\\\vdots\\0\\1\\0\\\vdots\\0\end{pmatrix}$

<u>$\lambda = -1$</u>: $\begin{pmatrix}1\\0\\\vdots\\0\\-1\end{pmatrix}, \begin{pmatrix}0\\1\\0\\\vdots\\0\\-1\\0\end{pmatrix}, \dots, \begin{pmatrix}0\\\vdots\\0\\1\\-1\\0\\\vdots\\0\end{pmatrix}$ or $\begin{pmatrix}0\\\vdots\\0\\1\\0\\-1\\0\\\vdots\\0\end{pmatrix}$

(36) Write $A = S^{-1} D S$, where D is diagonal.
Since $A^2 = (S^{-1} D S)^2 = S^{-1} D (S S^{-1}) D S$
$= S^{-1} D^2 S$,

$$S^{-1} D S = A = A^2 = S^{-1} D^2 S \implies D^2 = D.$$

Thus the eigenvalues must satisfy $\lambda^2 = \lambda$, that is, $\lambda = 0$ or 1. This is characteristic of an orthogonal projection — D (and hence A) is the identity on a subspace of $\mathbb{R}^n$, and vanishes on its orthogonal complement.