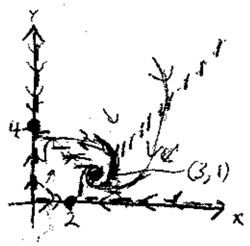
8.3

1) a)



$\frac{dx}{dt} = 0$ if $x = 0$

$y = x - 2$

$\frac{dy}{dt} = 0$ if $y = 0$

$y = 4 - x$

b) $J = \begin{bmatrix} -3 & 3 \\ -1 & -1 \end{bmatrix}$

$\frac{\partial f}{\partial x} = 2 - 2x + y \qquad \frac{\partial g}{\partial x} = -y$

$\frac{\partial f}{\partial y} = x \qquad \frac{\partial g}{\partial y} = 4 - 2y - x$

c) $\lambda^2 + 4\lambda + 6 \Rightarrow \lambda = \frac{-4 \pm \sqrt{16 - 24}}{2} = -2 \pm i\sqrt{2}$

$\boxed{\text{stable}}$ (real part < 0)

2) a) $\left.\begin{array}{l} \frac{dx}{dt} = \frac{dy}{dt} = 0 \\ y \neq 0 ; x \neq 0 \end{array}\right\} \Rightarrow 1 - x + ky - k = 1 - y + kx - k = 0$

$\Rightarrow x = 1 + ky - k$

$\Rightarrow 1 - y + k(1 + ky - k) - k = 0$

$\Rightarrow 1 - y + \cancel{k} + k^2 y - k^2 - \cancel{k} = 0$

$y(k^2 - 1) = (k^2 - 1) \Rightarrow \boxed{\begin{array}{l} y = 1 \\ x = 1 \end{array}}$

b) $\frac{\partial f}{\partial x} = 1 - 2x + ky - k \qquad \frac{\partial g}{\partial x} = ky$

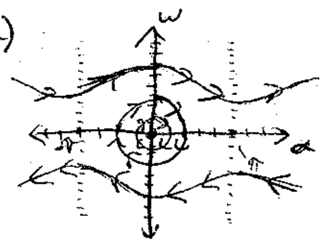
$\frac{\partial f}{\partial y} = kx \qquad \frac{\partial g}{\partial y} = 1 - 2y + kx - k$

$\Rightarrow J = \begin{bmatrix} -1 & k \\ k & -1 \end{bmatrix}$

2) c) $\lambda^2 + 2\lambda + (1-k^2) = 0 \Rightarrow \lambda = \frac{-2 \pm \sqrt{4-4+4k^2}}{2} = -1 \pm k$

if $-1 \pm k < 0$ then stable $\Rightarrow \boxed{-1<k<1}$

3) a)



b) equilibrium pts

at $w=0$, $\alpha = 0, 2\pi, 4\pi, \ldots = 2k\pi;\ k \in \mathbb{Z}$

or $w=0$, $\alpha = \pi, 3\pi, 5\pi, \ldots = (2k+1)\pi$

$\frac{\partial f}{\partial \alpha} = 0 \qquad \frac{\partial f}{\partial w} = 1$

$\frac{\partial g}{\partial \alpha} = \frac{-g}{L}\cos\alpha \qquad \frac{\partial g}{\partial w} = 0$

$\Rightarrow J = \begin{bmatrix} 0 & 1 \\ \frac{-g}{L}\cos\alpha & 0 \end{bmatrix} \Rightarrow \lambda^2 = -\frac{g}{L}\cos\alpha$

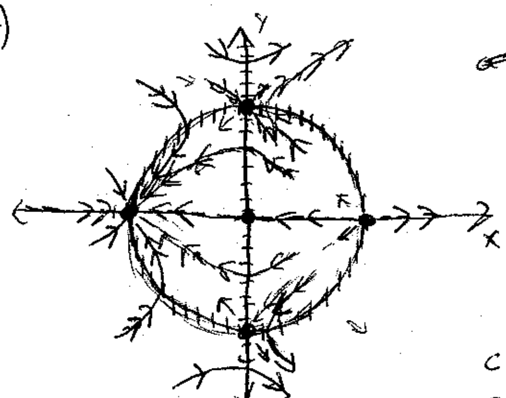
if $\alpha = 2k\pi;\ k \in \mathbb{Z}$

then $\lambda = \pm i\sqrt{\frac{g}{L}}$ & trajectories circle the equilibrium pt.

if $\alpha = (2k+1)\pi$

then $\lambda = \pm\sqrt{\frac{g}{L}}$ & trajectories deviate from equilibrium pt.

4)



$\frac{dx}{dt} = 0$ if $x^2 + y^2 = 1$

$\frac{dy}{dt} = 0$ if $x = 0$, $y = 0$

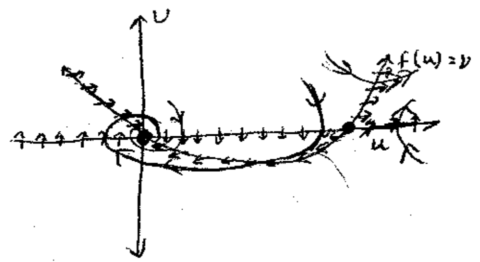
equilibria at $(1,0); (0,-1); (0,1); (-1,0)$

clearly $(-1,0)$ is the only stable equilibria (from graph).

5) $\begin{cases} \dfrac{du}{dt} = \nu \\ \dfrac{d\nu}{dt} = f(u) - \nu \end{cases}$ $\qquad \dfrac{du}{dt} = 0$ if $\nu = 0$

$\qquad\qquad\qquad\qquad\qquad \dfrac{d\nu}{dt} = 0$ if $f(u) = \nu$

graph nullclines



equilibria at $(0,0)$

and at $(1,0)$

find Jacobian: $\dfrac{\partial(\nu)}{\partial u} = 0 \qquad \dfrac{\partial(\nu)}{\partial \nu} = 1$

$$\frac{\partial(f(u)-\nu)}{\partial u} = f'(u) \qquad \frac{\partial(f(u)-\nu)}{\partial \nu} = -1$$

$$\Rightarrow J = \begin{bmatrix} 0 & 1 \\ f(u) & -1 \end{bmatrix} \quad \Rightarrow \lambda^2 + \lambda - f'(u) = 0$$

$$\Rightarrow \lambda_\pm = \frac{-1 \pm \sqrt{1+4f'(u)}}{2}$$

at $(1,0)$

$$f'(u)\big|_{u=1} > 0 \quad \Rightarrow \quad \lambda_+ = \frac{-1 + \sqrt{1+4f'(u)}}{2} > 0 \quad \Rightarrow \underline{\text{not stable}}$$

at $(0,0)$

$$f'(u)\big|_{u=0} < 0 \quad \Rightarrow \quad \lambda_+ = \frac{-1+\sqrt{1+4f'(u)}}{2} < 0 \quad \Rightarrow \underline{\text{stable}}$$

note: if $f'(u)\big|_{u=0} < -\frac{1}{4}$ then $\lambda_\pm$ are complex

& $Re[\lambda_\pm] = -\frac{1}{2} < 0 \quad \Rightarrow$ stable

from the graph $f(u)$ it looks as though the complex case is correct, however, either way $(0,0)$ is a stabe equilibrium point.