Math 21b Solution Set Section 9.3

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Question 4

Part A

This formula is just the dot product for vectors in $\mathbb{R}^{m\times 1} = \mathbb{R}^m$. Let $A = \begin{bmatrix} a_1 \\ \vdots \\ a_m \end{bmatrix}$ and $B = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$. Then $\langle A, B\rangle = \text{trace}(A^T B) = \text{trace}(A \cdot B) = \sum_{i=1}^m a_i b_i$.

Part B

This inner product is just like a "dot product":

$$A^T B = \begin{bmatrix} a_1 b_1 & \dots & \dots & a_1 b_n \\ a_2 b_1 & a_2 b_2 & \dots & a_2 b_n \\ \vdots & \vdots & \ddots & \vdots \\ a_n b_1 & \dots & \dots & a_n b_n \end{bmatrix} \Rightarrow \text{trace}(A^T B) = \sum_{i=1}^n a_i b_i$$

Question 8

By Fact 9.3.1 we see immediately that this transformation is linear.

The image of T(v) = ⟨v,w ⟩is the set of all real numbers $\mathbb{R}$, if $w \neq 0$. If $w = 0$ then the image is 0. One way of seeing this is to note that $\langle w, w\rangle = K > 0$, and thus by Fact 9.3.1 $\langle \frac{C}{K} w, w\rangle = C$ for all real numbers C.

Question 10

First, we identify the subset of P_2 orthogonal to t: for arbitrary g(t)

$$\int_{-1}^{1} t\,\mathrm{g}(t)dt = \int_{-1}^{1} t(at^2 + bt + c)dt = (1/4)at^4 + (1/3)bt^3 + (1/2)ct^2|_{-1}^{1} = (2/3)b$$

Therefore, we see that $b = 0 \Rightarrow \langle t, (at^2 + bt + c)\rangle = 0$. A non-orthogonal basis of this subspace would be $\{t^2, 1\}$. Using Gram-Schmidt, we find

$$w_1 = \sqrt{5}t^2$$

since

$$\|t^2\| = \sqrt{\langle t^2, t^2\rangle} = \frac{1}{\sqrt{5}}$$

Now using the next step of Gram Schmidt, we can write the formula for w_2:

$$w_2 = \frac{1}{\|(1 - \langle 1, w_1\rangle w_1)\|}(1 - \langle 1, w_1\rangle w_1)$$

Calculating the inner product gives us

$$\langle 1, w_1\rangle = \frac{1}{2}\int_{-1}^{1} \sqrt{5}t^2 dt = \frac{\sqrt{5}}{3}$$

and then finding component of 1 orthogonal to w_1 gives us

$$1 - \langle 1, w_1\rangle w_1 = 1 - \frac{5}{3}t^2$$

so that we have for the

$$\begin{aligned}
\|1 - \langle 1, w_1\rangle w_1\| &= \sqrt{\frac{1}{2}\int_{-1}^{1}(1 - \frac{5}{3}t^2)^2 dt} \\
&= \sqrt{\frac{1}{2}\int_{-1}^{1} 1 - \frac{10}{3}t^2 + \frac{25}{9}t^4 dt} \\
&= \sqrt{\frac{1}{2}(t - \frac{10}{9}t^3 + \frac{5}{9}t^5|_{-1}^{1}} \\
&= \sqrt{1 - \frac{10}{9} + \frac{5}{9}} \\
&= \sqrt{\frac{4}{9}} \\
&= \frac{2}{3}
\end{aligned}$$

So we have

$$w_2 = \frac{3}{2}(1 - \frac{5}{3}t^2)$$

Question 12

We know that $\mathrm{f}(t) = |t|$. Using the formulas form Fact 9.3.5, we can find all the Fourier coefficients using the definition of the inner product and integration:

$$
\begin{aligned}
a_0 &= \frac{1}{\sqrt{2}\pi}\int_{-\pi}^{\pi} \mathrm{f}(t)dt \\
&= \frac{1}{\sqrt{2}\pi}\int_{0}^{\pi} tdt \frac{1}{\sqrt{2}\pi}\int_{-\pi}^{0} tdt \\
&= \frac{1}{\sqrt{2}\pi}(\frac{1}{2}\pi^2 + \frac{1}{2}\pi^2) \\
&= \frac{\pi}{2\sqrt{2}}
\end{aligned}
$$

Immediately, since $\mathrm{f}(t)$ is even, we know that $b_k = 0 \forall k$. Using the definition, we find the same value:

$$
\begin{aligned}
b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} |t|\sin(kt)dt \\
&= \frac{1}{\pi}\int_{0}^{\pi} t\sin(kt)dt + \frac{1}{\pi}\int_{-\pi}^{0} (-t)\sin(kt)dt \\
&= \frac{1}{\pi}\int_{0}^{\pi} t\sin(kt)dt - \frac{1}{\pi}\int_{0}^{\pi} t\sin(kt)dt \\
&= 0
\end{aligned}
$$

Now, for Fourier coefficients c_k, notice that the integrand is an even function:

$$
\begin{aligned}
c_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} |t|\cos(kt)dt \\
&= \frac{2}{\pi}\int_{0}^{\pi} t\cos(kt)dt \\
&= \frac{2}{\pi}[\frac{t}{k}\sin(kt) + \frac{1}{k^2}\cos(kt)]_0^{\pi} \\
&= 0 \text{ if k is even} \\
&= \frac{-4}{k^2\pi} \text{ if k is odd}
\end{aligned}
$$

Question 26

$$
\begin{aligned}
a_0 &= \frac{1}{\sqrt{2}\pi}\int_{-\pi}^{\pi} \mathrm{f}(t)dt \\
&= \frac{1}{\sqrt{2}\pi}[\int_{0}^{\pi} dt + \int_{0}^{-\pi} (-1)dt] \\
&= \frac{1}{\sqrt{2}\pi}[\pi - \pi] \\
&= 0
\end{aligned}
$$

The product of two odd functions is an even function, so the integrand we use to find b_k is even, and thus we need only to examine its integral on half the interval$[-\pi, pi]$:

$$\begin{aligned} b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} \mathrm{f}(t)\sin(kt)dt \\ &= \frac{2}{\pi}\int_{0}^{\pi} \mathrm{f}(t)\sin(kt)dt \\ &= \frac{2}{\pi}\int_{0}^{\pi} \sin(kt)dt \\ &= \frac{2}{\pi}[\frac{-1}{k}\cos(kt)]_0^\pi \\ &= 0 \text{ if k is even} \\ &= \frac{4}{k\pi} \text{ if k is odd} \end{aligned}$$

$c_k = 0(\forall k)$ since $\mathrm{f}(t)$ is odd.

Question 28

From Fact 9.3.6, we know that

$$a_0 + \sum_{i=1}^{\infty} b_i^2 + \sum_{j=1}^{\infty} c_j^2 = \|f\|^2 = \langle f, f\rangle$$

Using our answers from Question 26, we can find the value of $\|f\|^2$ and then make deductions about the values of the infinite series that we obtain from the Fourier coefficients. In particular:

$$\langle f, f\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} (\mathrm{f}(t))^2 dt = \frac{1}{\pi}\int_{-\pi}^{\pi} (1)dt = 2$$

Using this and the value of the Fourier coefficients from question 26, we see that

$$\sum_{k \text{ is odd}} (\frac{4}{k\pi})^2 = \sum_{k \text{ is odd}} (\frac{16}{k^2\pi^2}) = 2 \Rightarrow \sum_{j=0}^{\infty} \frac{1}{(2j+1)^2} = \frac{\pi^2}{8}$$