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16. Follow example 7, and fact 9.4.10. Our basis will be $\cos(t), \sin(t)$. Then the matrix, A, with respect to this basis is $\begin{bmatrix} 12 & 4 \\ -4 & 12 \end{bmatrix}$. Now, working in our basis, the differential equation has the form $A\vec{x} = \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$, with the solution $\vec{x} = A^{-1}\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right] = \frac{1}{160}\begin{bmatrix} 12 & -4 \\ 4 & 12 \end{bmatrix}\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right] = \left[\begin{smallmatrix}3/40\\1/40\end{smallmatrix}\right]$, so we have a solution of

$$\frac{3}{40}\cos(t) + \frac{1}{40}\sin(t).$$

Another way to do this is to say $f = P\cos(t) + Q\sin(t)$, so $-P + 4Q + 13P = 1$, and $-Q - 4P + 13Q = 0$, and thus $P = 3Q$, so $Q = \frac{1}{40}$ and $P = \frac{3}{40}$. Now that we have a solution to the inhomogenous equation, we must solve the homogenous equation. The characteristic polynomial is $\lambda^2 + 4\lambda + 13$, which has roots $\lambda = -2 \pm \sqrt{4-13} = -2 \pm 3i$. By fact 9.4.9, we know that the general solution to functions in the kernel is $e^{-2t}(c_1\cos(3t) + c_2\sin(3t))$. Thus, the complete solution to our differential equation is

$$e^{-2t}(c_1\cos(3t) + c_2\sin(3t)) + \frac{-3}{40}\cos(t) + \frac{1}{40}\sin(t)$$

18. This is done the same way. Our inhomogenous solution has coefficients satisfying $P+3Q = 1$, $Q-3P = 0$, so $P = \frac{1}{10}$, and $Q = \frac{3}{10}$. The homogenous solution is of the form $c_1e^{-t} + c_2e^{-2t}$ (the roots of the char. poly. are -1 and -2), so the solution is

$$c_1e^{-t} + c_2e^{-2t} + \frac{1}{10}\cos(t) + \frac{3}{10}\sin(t)$$

30.a. The rate of change of temperature is dependent on a few things. First, there is the current temperature: $T(t)$. Then there is the ambient temperature, A. If there is no difference between the two, then there will be no change, and the sign of the change is the negative of the difference between $T(t)$ and A. Finally, there is the rate at which temperature is exchanged – the conductivity of the coffee cup, the rate of loss off the top, etc. That is k.

b. Rewrite this equation as $T'(t) + kT(t) = kA$. Looking at the hint, we notice that a trivial solution is $T(t) = A$, since A is just a constant – the outside temperature. Then we can solve for the homogenous solution, which is e^{-kt}. Thus, the solution is

$$c_1e^{-kt} + A.$$

This makes sense because we expect some kind of inverse exponential heat loss, as well as a base temperature which cannot dip below the ambient temperature.

38.a. $(D-\lambda)\left(p(t)e^{\lambda t}\right) = p'(t)e^{\lambda t} + \lambda p(t)e^{\lambda t} - \lambda p(t)e^{\lambda t} = p'(t)e^{\lambda t}$

b. By a), we know $(D-\lambda)^m\left(p(t)e^{\lambda t}\right)$ must be 0, since if the degree of p is k, after $(D-\lambda)^k$, $p^{(k)}$ is constant, and the next application of $(D-\lambda)$ makes it 0.

c. Since $(D-\lambda)^m$ is of m^th degree, it has m basis elements. These elements are $1\times e^{\lambda t}, te^{\lambda t}, \ldots, t^{m-1}e^{\lambda t}$.

d. Just combine the bases of each component of this operator. Thus, a basis is $e^{\lambda_1 t}, te^{\lambda_1 t}, \ldots, t^{m_1-1}e^{\lambda_1 t}, e^{\lambda_2 t}, te^{\lambda_2 t}, \ldots, t^{m_r-1}e^{\lambda_r t}$.

40. Factoring this DE, we get $(D-1)(D+1)^2$, so our basis is e^t, e^{-t}, te^{-t}, so the solutions are of the form

$$c_1e^t + c_2e^{-t} + c_3te^{-t}.$$

46. (Note the typo – they left out an "$\vec{x}$"). $x_3'(t) = x_3(t)$, so $x_3(t) = c_3e^t = -1e^t$. That's good. $x_2'(t) = x_2(t) + 2x_3(t) = x_2(t) - 2e^t$, so $x_2'(t) - x_2(t) = -2e^t$. The homogenous solution is e^t. For the inhomogenous one, use Fact 9.4.13 to get $x_2(t) = e^t\int -2e^{-t}e^t dt = -2te^t$. Thus, $x_2(t) = c_2e^t - 2te^t$, and so $x_2(t) = e^t - 2te^t$ (look at initial conditions). Finally, we have that $x_1'(t) = 2x_1(t) + 3e^t - 6te^t - e^t = 2x_1(t) + 2e^t - 6te^t$. $x_1'(t) - 2x_1(t) = 2e^t - 6te^t$. The homogenous solution is e^{2t}, and the inhomogenous one $e^{2t}\int e^{-2t}(2e^t - 6te^t)\,dt = e^{2t}\int 2e^{-t} - 6te^{-t}dt = -2e^t + 6te^t + 6e^t = 4e^t + 6te^t$, so $x_1(t) = -2e^{2t} + 4e^t + 6te^t$, and we have

$$\vec{x} = e^t\begin{bmatrix} -2e^t + 4 + 6t \\ 1 - 2t \\ -1 \end{bmatrix}$$