Further Topics in Differential Equations

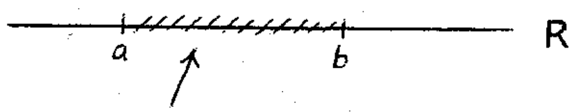
Richard Taylor

April 25, 1999

1 Fourier Series

We have looked at spaces of functions which behaved like $\mathbf{R}^n$, but we did not look at any analogues of the concepts of length, angle or dot product. In this section we will discuss an example in which the analogues of these concepts play an important role.

Recall that if a and b are real numbers with $a < b$, then $[a, b]$ denotes the interval $\{x \in \mathbf{R} : a \leq x \leq b\}$.



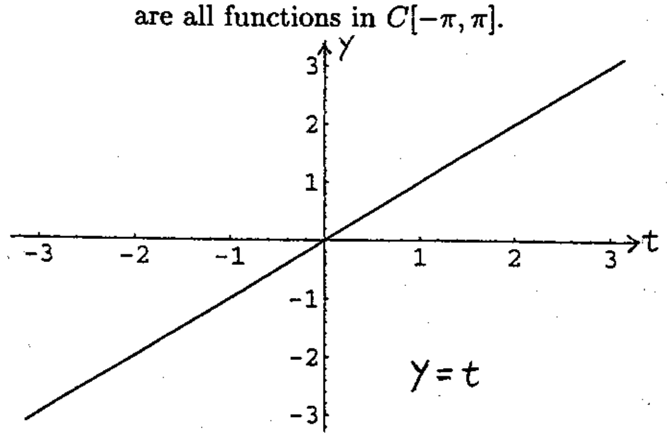
interval $[a, b]$ (includes end points)

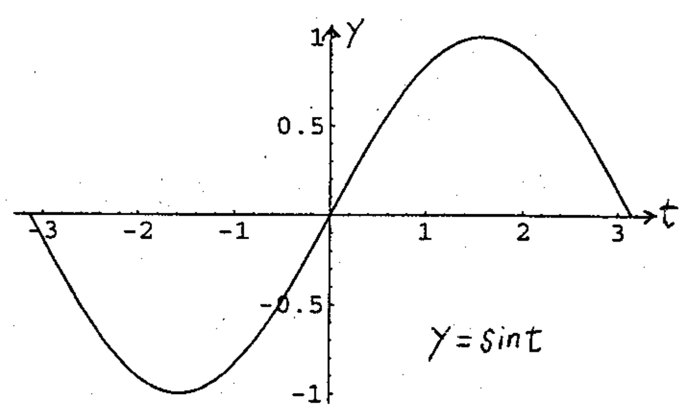
We will let $C[-\pi, \pi]$ denote the collection of all continuous functions from the interval $[-\pi, \pi]$ to $\mathbf{R}$.

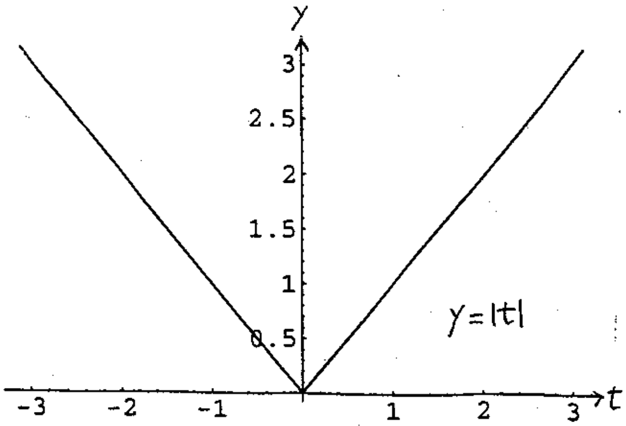
For example,

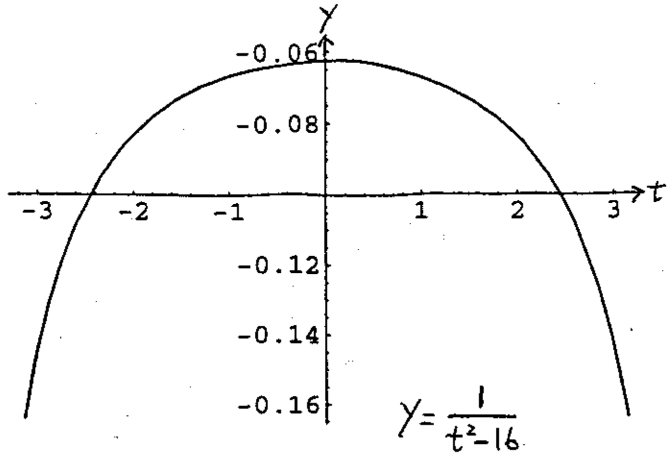
$$t, \sin t, |t|, \frac{1}{t^2 - 16}$$

are all functions in $C[-\pi, \pi]$.





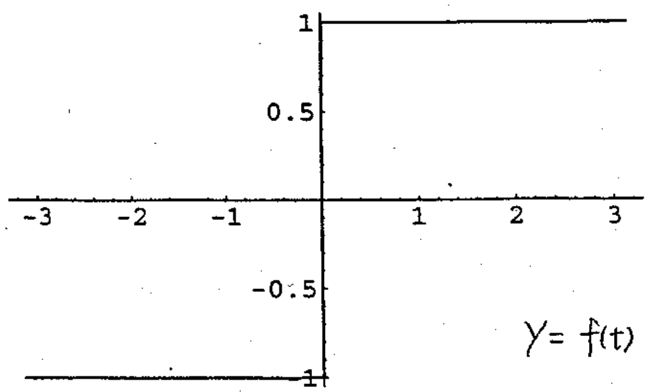




On the other hand, the function

$$f(t) = \begin{cases} 1 & t > 0 \\ 0 & t = 0 \\ -1 & t < 0 \end{cases}$$

does not lie in $C[-\pi, \pi]$.



Again $C[-\pi, \pi]$ is a linear space:

(a) If $f(t)$ and $g(t) \in C[-\pi, \pi]$, then $f(t) + g(t) \in C[-\pi, \pi]$. (Recall that the sum of continuous functions is continuous.)
e.g. $|t| + \sin t \in C[-\pi, \pi]$.

(b) If $f(t) \in C[-\pi, \pi]$ and $c \in \mathbf{R}$, then $cf(t) \in C[-\pi, \pi]$.
e.g. $2 \sin t \in C[-\pi, \pi]$.

We will define the *inner product* of two functions $f(t), g(t) \in C[-\pi, \pi]$ to be

$$\langle f(t), g(t) \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) g(t)\, dt$$

You should think of it as an analogue of the dot product of two vectors in $\mathbf{R}^n$.

It shares with the dot product the following three key properties:

(1) *If* $f(t)$ *and* $g(t) \in C[-\pi, \pi]$, *then*

$$\langle f(t), g(t)\rangle = \langle g(t), f(t)\rangle$$

(2) *If* $f(t), g(t)$ *and* $h(t) \in C[-\pi, \pi]$, *and if* $c \in \mathbf{R}$, *then*

$$\langle c_1 f(t) + c_2 g(t), h(t)\rangle = c_1\langle f(t), h(t)\rangle + c_2\langle g(t), h(t)\rangle$$ (Equivalent, but better expressed.)

$$\langle cf(t) + g(t), h(t)\rangle = c\,\langle f(t), h(t)\rangle + \langle g(t), h(t)\rangle$$

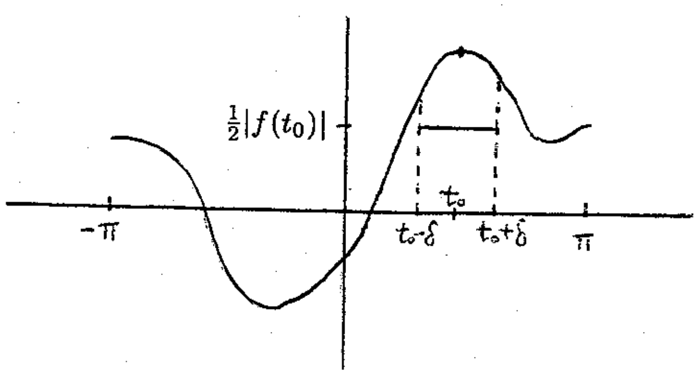
(3) *If* $f(t) \in C[-\pi, \pi]$ *is non-zero (i.e., not identically zero), then*

$$\langle f(t), f(t)\rangle > 0$$

We will let you check properties (1) and (2) for yourself. Let us explain property (3). Firstly,

$$\langle f(t), f(t)\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)^2\,dt$$

As $f(t)^2 \geq 0$ for all t, we see that $\langle f(t), f(t)\rangle \geq 0$. Suppose $f(t) \neq 0$, why is $\langle f(t), f(t)\rangle \neq 0$? Well, suppose $f(t_0) \neq 0$. Because f is continuous, we can find $\delta > 0$ such that $|f(t)| > \frac{1}{2}|f(t_0)|$ for all $t \in [t_0 - \delta, t_0 + \delta]$. As long as $t_0 \neq \pm\pi$ we may also suppose $t_0 - \delta > -\pi$ and $t_0 + \delta < \pi$. (We leave the cases $t_0 = \pm\pi$ to you. They are only slightly different.)

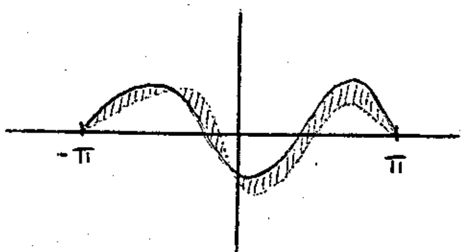


Then

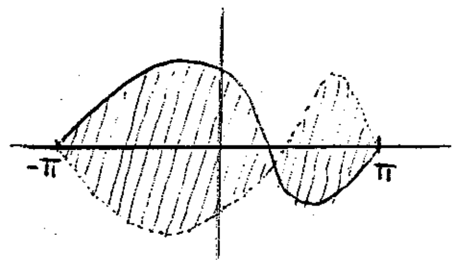
$$\int_{-\pi}^{\pi} f(t)^2 dt \;\; \geq \;\; \int_{t_0-\delta}^{t_0+\delta} f(t)^2 dt$$

$$\begin{aligned}
&\geq \int_{t_0-\delta}^{t_0+\delta} \frac{1}{4} f(t_0)^2 dt \\
&= \frac{\delta}{2}|f(t_0)|^2 \\
&> 0
\end{aligned}$$

We define the *length* (or "norm") of function $f(t) \in C[-\pi, \pi]$ to be $\sqrt{\langle f(t), f(t) \rangle}$, and we will denote it $\| f \|$. We define the *distance* between two functions $f(t), g(t) \in C[-\pi, \pi]$ to be $\| f - g \|$. Roughly speaking, two functions $f(t)$ and $g(t)$ are close if the area between their graphs is small.



close functions



far apart functions

Examples

(1) *Calculate* $\| t \|$.

$$\begin{aligned}
\langle t, t \rangle &= \frac{1}{\pi} \int_{-\pi}^{\pi} t^2\, dt = \frac{1}{\pi} \left[\frac{t^3}{3} \right]_{-\pi}^{\pi} = \frac{2\pi^3}{3\pi} = \frac{2\pi^2}{3} \\
\| t \| &= \sqrt{\frac{2}{3}}\pi
\end{aligned}$$

(2) *Calculate the distance between* 1 *and* $|t|$.

$$\begin{aligned}
\langle |t| - 1, |t| - 1 \rangle &= \frac{1}{\pi} \int_{-\pi}^{\pi} (|t| - 1)^2\, dt \\
&= \frac{2}{\pi} \int_0^{\pi} (t-1)^2 dt \\
&= \frac{2}{\pi} \int_0^{\pi} (t^2 - 2t + 1) dt \\
&= \frac{2}{\pi} \left[\frac{t^3}{3} - t^2 + t \right]_0^{\pi}
\end{aligned}$$

$$= \frac{2}{3}\pi^2 - 2\pi + 2$$

$$\| \, |t| - 1 \, \| = \sqrt{\frac{2}{3}\pi^2 - 2\pi + 2}$$

(3) *If n is a positive integer, find $\| \sin nt \|$.*

$$\langle \sin nt, \sin nt \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} (\sin nt)^2 \, dt$$

To evaluate this integral, recall the useful trigonometric formulae:

$$\begin{aligned} \sin(A+B) &= \sin A \cos B + \cos A \sin B \\ \cos(A+B) &= \cos A \cos B - \sin A \sin B \\ 1 &= (\cos A)^2 + (\sin A)^2 \end{aligned}$$

Putting $B = A$ in the second of these we get

$$\cos(2A) = (\cos A)^2 - (\sin A)^2 = 1 - 2(\sin A)^2$$

Thus

$$\langle \sin nt, \sin nt \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{1}{2}(1 - \cos 2nt) \, dt = 1 - \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos 2nt \, dt = 1$$

Thus $\| \sin nt \| = 1$.

Similarly, if n is a positive integer one can check that

$$\| \cos nt \| = 1$$

Moreover,

$$\| \frac{1}{\sqrt{2}} \| = 1$$

We will call two fuctions $f(t), g(t) \in C[-\pi, \pi]$ *orthogonal* if $\langle f(t), g(t) \rangle = 0$.

We will call a (finite or infinite) collection of functions $f_1(t), \cdots, f_n(t), \cdots$ *othornormal* if

(a) $\| f_j(t) \| = 1$ for each j

(b) $\langle f_j(t), f_k(t) \rangle = 0$ if $j \neq k$

Examples

(1) If $n \neq m$ are positive integers, then $\sin nx$ and $\sin mx$ are orthogonal.

$$\begin{aligned}\langle \sin nt, \sin mt\rangle &= \frac{1}{\pi}\int_{-\pi}^{\pi} \sin nt\ \sin mt\, dt \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{1}{2}(\cos(n-m)t - \cos(n+m)t)\, dt \\ &= \frac{1}{2\pi}\left[\frac{\sin(n-m)t}{n-m} - \frac{\sin(n+m)t}{n+m}\right]_{-\pi}^{\pi} \\ &= 0\end{aligned}$$

Again we use the formula for $\cos(A+B)$ (and for $\cos(A-B)$).

(2) In fact the sequence of functions

$$\frac{1}{\sqrt{2}}, \sin t, \cos t, \sin 2t, \cos 2t, \sin 3t, \cos 3t, \cdots$$

is orthonormal. We leave it to you to evaluate the necessary integrals.

The following facts can be proved exactly as they were for $\mathbf{R}^n$.

(1) *If $f_1(t), \cdots, f_n(t)$ are orthonormal, then they form a basis of an n-dimensional subspace.*

The main point here is to check that $f_1(t), \cdots, f_n(t)$ are linearly independent. Suppose

$$c_1 f_1(t) + \cdots + c_n f_n(t) = 0$$

Taking the inner product

$$\langle f_j(t), c_1 f_1(t) + \cdots + c_n f_n(t)\rangle = 0$$

We see that

$$0 = c_1 \langle f_j(t), f_1(t)\rangle + \cdots + c_n \langle f_j(t), f_n(t)\rangle = c_j$$

for each j.

(2) *If $f(t)$ and $g(t)$ are orthogonal, then*

$$\| f(t) + g(t) \|^2 = \| f(t) \|^2 + \| g(t) \|^2$$

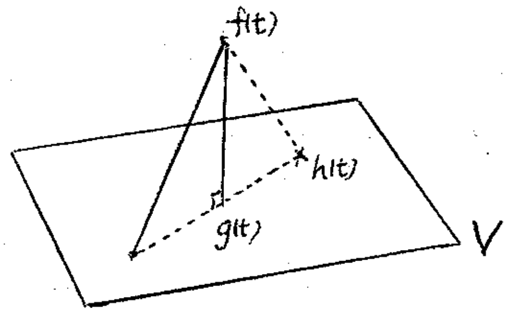
Indeed

$$\begin{aligned} \| f+g \|^2 &= \langle f+g, f+g\rangle \\ &= \langle f,f\rangle + \langle f,g\rangle + \langle g,f\rangle + \langle g,g\rangle \\ &= \langle f,f\rangle + \langle g,g\rangle \\ &= \| f \|^2 + \| g \|^2 \end{aligned}$$

(3) *Suppose V is a subspace of $C[-\pi,\pi]$ and that $f(t) \in C[-\pi,\pi]$. If we can find $g(t) \in V$ such that $f(t)-g(t)$ is orthogonal to each element of V, then*

$$\| f(t)-g(t) \| \le \| f(t)-h(t) \|$$

for all $h(t) \in V$, with equality if and only if $g(t) = h(t)$.



We have

$$\begin{aligned} \| f(t)-h(t) \|^2 &= \| f(t)-g(t)+g(t)-h(t) \|^2 \\ &= \| f(t)-g(t) \|^2 + \| g(t)-h(t) \|^2 \\ &\geqslant \| f(t)-g(t) \|^2 \end{aligned}$$

with equality if and only if $\| g(t)-h(t) \|^2 = 0$, i.e., $g(t) = h(t)$.
The main point is that $g(t)-h(t)$ is in V and so orthogonal to $f(t) - g(t)$.

(4) *If $f_1(t), \cdots, f_n(t)$ are an orthonormal basis of a subspace $V \subset C[-\pi,\pi]$, then*

$$proj_V(f(t)) = \langle f(t), f_1(t)\rangle f_1(t) + \cdots + \langle f(t), f_n(t)\rangle f_n(t)$$

is in V, $f(t) - proj_V(f(t))$ is orthogonal to every element of V, and $proj_V(f(t))$ is closer to $f(t)$ than any other element of V.

It suffices to check that for each $j = 1, \cdots, n$,

$$\langle f_j(t), proj_V(f(t)) - f(t) \rangle = 0$$

But

$$\begin{aligned} & \langle f_j(t), proj_V(f(t)) - f(t) \rangle \\ = \; & \langle f(t), f_1(t) \rangle \langle f_j(t), f_1(t) \rangle + \cdots + \langle f(t), f_n(t) \rangle \langle f_j(t), f_n(t) \rangle - \langle f_j(t), f(t) \rangle \\ = \; & \langle f(t), f_j(t) \rangle - \langle f_j(t), f(t) \rangle \\ = \; & 0 \end{aligned}$$

We will let T_n denote the subspace of $C[-\pi, \pi]$ with orthonormal basis

$$\frac{1}{\sqrt{2}}, \sin t, \cos t, \cdots, \sin nt, \cos nt$$

Then $proj_{T_n}(f(t))$ is an approximation to $f(t)$ constructed from these trigonometric functions. As n increases one might expect these approximations become better and better. In fact we have:

Fact 1.1 (1) *If $f \in C[-\pi, \pi]$, then $\| proj_{T_n}(f(t)) - f(t) \| \to 0$ as $n \to \infty$.*
(2)

$$\| f(t) \|^2 = \left\langle f(t), \frac{1}{\sqrt{2}} \right\rangle^2 + \sum_{n=1}^{\infty} \left(\langle f(t), \sin nt \rangle^2 + \langle f(t), \cos nt \rangle^2 \right)$$

Although we will not prove this , let us at least explain how (2) follows from (1).

$$\begin{aligned} & \| proj_{T_n}(f(t)) \| \\ = \; & \| \left\langle f(t), \frac{1}{\sqrt{2}} \right\rangle \frac{1}{\sqrt{2}} + \langle f(t), \sin t \rangle \sin t + \cdots + \langle f(t), \cos nt \rangle \cos nt \|^2 \\ = \; & \left\langle f(t), \frac{1}{\sqrt{2}} \right\rangle^2 + \langle f(t), \sin t \rangle^2 + \cdots + \langle f(t), \cos nt \rangle^2 \end{aligned}$$

On the other hand,

$$\begin{aligned} \| f(t) \|^2 &= \| proj_{T_n}(f(t)) \|^2 + \| f(t) - proj_{T_n}(f(t)) \|^2 \\ &= \left\langle f(t), \frac{1}{\sqrt{2}} \right\rangle^2 + \cdots + \langle f(t), \cos nt \rangle^2 + \| f(t) - proj_{T_n}(f(t)) \|^2 \end{aligned}$$

Letting $n \to \infty$ gives part (2).

Although this tells us that "on average" $proj_{T_n}(f(t))$ is close to $f(t)$, it does not tell us what happens for any given $t \in [-\pi, \pi]$. However, if we place some smoothness hypothesis on $f(t)$ then we can say what happens.

Fact 1.2 *Suppose $f(t) \in C[-\pi, \pi]$ is differentiable at a point $x \in C[-\pi, \pi]$, and if $x = \pm\pi$ also assume that $f(-\pi) = f(\pi)$. Then the series*

$$\left\langle f(t), \frac{1}{\sqrt{2}} \right\rangle \frac{1}{\sqrt{2}} + \langle f(t), \sin t \rangle \sin x + \cdots + \langle f(t), \cos nt \rangle \cos nx$$

converges to $f(x)$.

The series

$$\left\langle f(t), \frac{1}{\sqrt{2}} \right\rangle \frac{1}{\sqrt{2}} + \langle f(t), \sin t \rangle \sin x + \cdots + \langle f(t), \cos nt \rangle \cos nx$$

is called the *Fourier series for f* after the French mathematician Jean-Baptiste-Joseph Fourier (1768-1830). 1.2 was known to Fourier and is often referred to as Fourier's theorem, although the first rigorous proof was only found later by Dirichlet.

(*We recommend that you read pages 520 through 523 of Section 9.3*)

Example

Find the Fourier series for t.

$$\begin{aligned}
\left\langle \frac{1}{\sqrt{2}}, t \right\rangle &= \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{t}{\sqrt{2}}\, dt = 0 \\
\langle \cos nt, t \rangle &= \frac{1}{\pi} \int_{-\pi}^{\pi} t \cos nt\, dt = 0 \\
& \quad \text{because } t \cos nt \text{ is an odd funciton} \\
\langle \sin nt, t \rangle &= \frac{1}{\pi} \int_{-\pi}^{\pi} t \sin nt\, dt \\
&= \frac{1}{\pi} \left[\frac{-\cos nt}{n} t \right]_{-\pi}^{\pi} + \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{\cos nt}{n}\, dt \\
&= \frac{1}{\pi} \left(\frac{-(-1)^n}{n} \pi - \frac{-(-1)^n}{n} (-\pi) \right) \\
&= \frac{2(-1)^{n+1}}{n}
\end{aligned}$$

Thus

$$t = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} sin\, nt$$

By part(2) of 1.1, we see that

$$\begin{aligned}
\| t \|^2 &= \sum_{n=1}^{\infty} \frac{4}{n^2} \\
\frac{2}{3}\pi^2 &= 4\sum_{n=1}^{\infty} \frac{1}{n^2} \\
\frac{\pi^2}{6} &= \sum_{n=1}^{\infty} \frac{1}{n^2} \\
&= 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36} + \cdots
\end{aligned}$$

an amazing expression for π as an infinite sum.

On the other hand, by 1.2, if we put $t = \frac{\pi}{2}$, we get

$$\begin{aligned}
\frac{\pi}{2} &= 2 \cdot \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{(-1)^{(n+1)}}{n} (-1)^{\frac{n-1}{2}} \\
&= 2 \cdot \sum_{m=0}^{\infty} \frac{(-1)^m}{2m+1} \\
&= 2\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots\right)
\end{aligned}$$

another amazing expression for π as an infinite sum.

Exercises

(1) Find the length of $1 + \sin t + 3\cos 5t + 2\sin 10t$.

(2) Show that $\frac{1}{\sqrt{2}}$ and $\cancel{\frac{t}{\pi}}$ are orthonormal. The second function should be $\sqrt{\frac{3}{2}} \cdot \frac{t}{\pi}$
Let V be the subspace of $C[-\pi, \pi]$ consisting of functions of the form $at + b$. Find $proj_V(t^2)$.

(3) Find the Fourier series for $|t|$.

(4) Calculate

$$\int_{-\pi}^{\pi} e^{at} \cos nt \, dt$$

HINT: Integrate by parts twice to get an expression

$$\int_{-\pi}^{\pi} e^{at}\cos nt\,dt = \left[\frac{a\cos nt}{n^2}e^{at}\right]_{-\pi}^{\pi} - \frac{a^2}{n^2}\int_{-\pi}^{\pi} e^{at}\cos nt\,dt$$

and then solve for

$$\int_{-\pi}^{\pi} e^{at}\cos nt\,dt$$

.

(5) If a is a real constant, find the Fourier series for

$$\cosh at = \frac{1}{2}(e^{at} + e^{-at})$$

HINT: $\cosh(-at) = \cosh(at)$.

(6) Find a closed formula for

$$\sum_{n=1}^{\infty} \frac{1}{n^2 + a^2}$$

as a function of a.

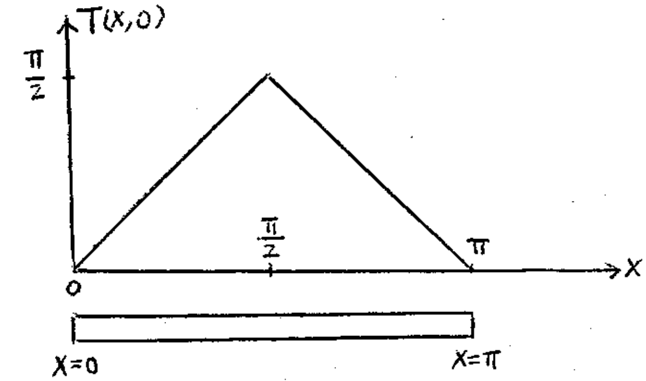
HINT: Use (5) and 1.2.

2 Partial Differential Equations I: The Heat Equation

Consider a uniform metal bar stretching from $x = 0$ to $x = \pi$. Suppose that the ends of the bar are held at a constant temprature of 0. (e.g. are immersed in a mixture of water and ice), but that otherwise the bar is thermally insulated from its surroundings, except that at time $t = 0$, the bar is quickly heated so that it has temperature distribution

$$T(x,0) = \begin{cases} x & \text{if } x \le \frac{\pi}{2} \\ \pi - x & \text{if } x \ge \frac{\pi}{2} \end{cases}$$

Describe the temperature of the bar at all subsequent times.



The temperature $T(x,t)$ obeys the equation:

$$\frac{\partial T}{\partial t} = \mu \frac{\partial^2 T}{\partial x^2}$$

for some positive constant μ depending on the structure of the bar. (This sort of equation is called a *partial differential equation* or *PDE*).

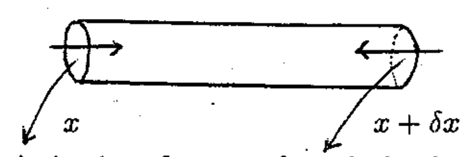
Where does this particular equation come from?

rate of heat flow post x is $-\kappa \frac{\partial T}{\partial x}$, κ =thermal conductivity

(heat flows from hot to cool at a rate proportional to the temperature gradient)

rate of temperature increase×length= c·rate of arrival of heat

We examine what happens to a small length of bar from x to $x + \delta x$ in the small time from t to $t + \delta t$



heat in in time δt

$$-\kappa \frac{\partial T}{\partial x}(x,t)\delta t$$

heat in in time δt

$$\kappa \frac{\partial T}{\partial x}(x+\delta x,t)\delta t$$

total heat in in time δt

$$\kappa \left(\frac{\partial T}{\partial x}(x+\delta x,t) - \frac{\partial T}{\partial x}(x,t) \right) \delta t$$

rise in temperature in time δt

$$T(x,t+\delta t) - T(x,t) \quad \approx \quad \frac{c}{\delta x} \kappa \left(\frac{\partial T}{\partial x}(x+\delta x,t) - \frac{\partial T}{\partial x}(x,t) \right) \delta t$$

$$
\begin{aligned}
\frac{1}{\delta t}\left(T(x,t+\delta t)-T(x,t)\right) &\approx c\kappa\frac{1}{\delta x}\left(\frac{\partial T}{\partial x}(x+\delta x,t)-\frac{\partial T}{\partial x}(x,t)\right)\\
\frac{\partial T}{\partial t} &= c\kappa\frac{\partial^2 T}{\partial x^2}
\end{aligned}
$$

The equation

$$\frac{\partial T}{\partial t}=\mu\frac{\partial^2 T}{\partial x^2}$$

is called the *heat* or *diffusion* equation. It arises in many physical situations where some diffusion process occurs, e.g., diffusion of pollutants in an aquifer, or of ions through a cell wall.

Here we are asked to find a solution to this equation subject to the restrictions that

$$T(0,t)=T(\pi,t)=0$$

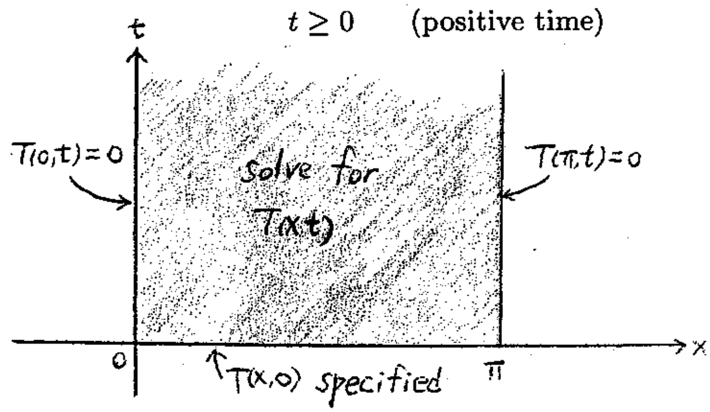
(the ends of the bar stay at temperature 0)

$$T(x,0)=\begin{cases} x & \text{if } x\le\frac{\pi}{2}\\ \pi-x & \text{if } x\ge\frac{\pi}{2}\end{cases}$$

Such restrictions are called *initial conditions* or *boundary conditions.* Many different initial conditions are possible, they will depend on the problem one is trying to solve. (Another possibility would be that the bar was initially at temperature 0, that the left hand end is always kept at temperature 0, but that the right hand end is made to take on a specified temperature $T(\pi,t)$.)

We are only looking for T in the region

$$
\begin{aligned}
0\le x\le\pi &\quad \text{(length of bar)}\\
t\ge 0 &\quad \text{(positive time)}
\end{aligned}
$$



There are several methods available to tackle this sort of problem. We will present one based on Fourier series.

We first look for some simple solutions to the equation:

$$\frac{\partial T}{\partial t} = \mu \frac{\partial^2 T}{\partial x^2} \qquad T(0,t) = T(\pi,t) = 0 \qquad (*)$$

In fact let us look for a solution:

$$T(x,t) = u(x)v(t)$$

Then we require:

$$u(0) = u(\pi) = 0 \text{ and } \frac{v'(t)}{v(t)} = \mu \frac{u''(x)}{u(x)}$$

We see that the quantity

$$\frac{v'(t)}{v(t)} = \mu \frac{u''(x)}{u(x)}$$

is independent of both position x and time t so that it must be a constant. We are led to try to solve the equations

$$\begin{aligned} u''(x) &= au(x) \\ v'(t) &= a\mu v(t) \\ u(0) = u(\pi) &= 0 \end{aligned}$$

But we know how to solve these equations.

(a) If $a > 0$, say $a = \lambda^2$, then

$$u(x) = Ae^{\lambda x} + Be^{-\lambda x}$$

The equations $u(0) = u(\pi) = 0$ imply that $A = B = 0$, i.e., $u(x) = 0$. This is not much help.

(b) If $a = 0$, then

$$u(x) = Ax + B$$

Again the equations $u(0) = u(\pi) = 0$ imply that $A = B = 0$, i.e., $u(x) = 0$. Again not much help.

(c) Now suppose $a < 0$, say $a = -\lambda^2$. Then

$$u(x) = A \sin \lambda x + B \cos \lambda x$$

The equation $u(0) = 0$ implies $B = 0$.
The equation $u(\pi) = 0$ implies $A = 0$ or λ is a whole number n. In this case $v\prime(t) = -n^2 \mu v(t)$ so that $v(t) = Ce^{-\mu n^2 t}$.

Thus we have found a series of solutions to $(*)$. Namely for each positive integer n we have a solution

$$c_n e^{-\mu n^2 t} sin\, nx$$

for some constant c_n.

If we put $t = 0$ we get $C_n \sin nx$ so none of these solutions is the one we are looking for.

However, note that both the equation

$$\frac{\partial T}{\partial t} = \mu \frac{\partial^2 T}{\partial x^2}$$

and the boundary conditions

$$T(0, t) = T(\pi, t)$$

are *linear*: i.e., if T_1 and T_2 are two solutions so is $cT_1 + T_2$. Thus we get a lot more solutions of these two equations: namely any finite sum

$$\sum_{n=1}^{N} c_n e^{-\mu n^2 t} \sin nx$$

At $t = 0$ this becomes

$$\sum_{n=1}^{N} c_n \sin nx$$

In fact, more is true. If the constants c_n become smaller sufficiently rapidly as $n \to \infty$ then the sum

$$\sum_{n=1}^{\infty} c_n e^{-\mu n^2 t} sin\, nx$$

will converge and give a solution to $(*)$ which specializes at $t = 0$ to

$$\sum_{n=1}^{\infty} c_n \sin nx$$

If we can find c_n such that

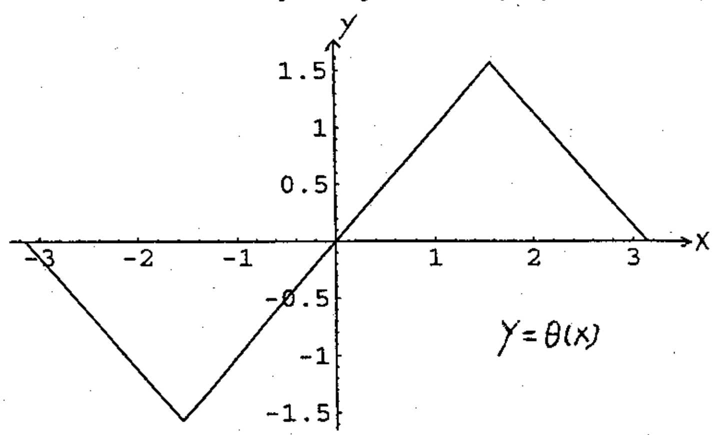
$$\sum_{n=1}^{\infty} c_n \sin nx = \begin{cases} x & x \leq \frac{\pi}{2} \\ \pi - x & x \geq \frac{\pi}{2} \end{cases}$$

then we would have found a solution to our original problem. But this is the sort of problem we studied in the last section.

To put it more precisely in the form we considered in the last section, consider

$$\theta(x) = \begin{cases} \pi - x & x \geq \frac{\pi}{2} \\ x & -\frac{\pi}{2} \leq x \leq \frac{\pi}{2} \\ -x - \pi & x \leq -\frac{\pi}{2} \end{cases}$$

Note that we extended θ to $[-\pi, \pi]$ by arranging that $\theta(-x) = -\theta(x)$.



We now compute the Fourier series of θ

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \theta(x)\frac{1}{\sqrt{2}}\,dx = 0 \quad \text{as } \theta(x) = -\theta(-x)$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \theta(x)\cos nx\,dx = 0 \quad \text{for the same reason.}$$

$$\begin{aligned}
\frac{1}{\pi}\int_{-\pi}^{\pi} \theta(x)\sin nx\,dx &= \frac{2}{\pi}\int_{0}^{\pi} \theta(x)\sin nx\,dx \\
&= \frac{2}{\pi}\int_{0}^{\frac{\pi}{2}} x\sin nx\,dx + \frac{2}{\pi}\int_{\frac{\pi}{2}}^{\pi} (\pi - x)\sin nx\,dx \\
&= \frac{2}{\pi}\int_{0}^{\frac{\pi}{2}} x\sin nx\,dx - \frac{2}{\pi}\int_{\frac{\pi}{2}}^{0} y\sin n(\pi - y)\,dy \\
&= \frac{2}{\pi}\int_{0}^{\frac{\pi}{2}} x\sin nx\,dx - \frac{2}{\pi}\int_{0}^{\frac{\pi}{2}} y\sin(ny - n\pi)\,dy \\
&= \frac{2}{\pi}(1 - (-1)^n)\int_{0}^{\frac{\pi}{2}} x\sin nx\,dx
\end{aligned}$$

$$
\begin{aligned}
&= \begin{cases} 0 & n \text{ even} \\ \frac{4}{\pi}\int_0^{\frac{\pi}{2}} x \sin nx\, dx & n \text{ odd} \end{cases} \\
\frac{4}{\pi}\int_0^{\frac{\pi}{2}} x \sin nx\, dx &= \frac{4}{\pi}\left[-\frac{\cos nx}{n}x\right]_0^{\frac{\pi}{2}} + \frac{4}{\pi}\int_0^{\frac{\pi}{2}} \frac{\cos nx}{n} dx \\
&= \frac{4}{\pi n^2}\left[\sin nx\right]_0^{\frac{\pi}{2}} \\
&= \frac{4}{\pi n^2}(-1)^{\frac{n-1}{2}} \qquad n \text{ odd} \\
\theta(x) &= \sum_{m=0}^{\infty} \frac{4(-1)^m}{\pi(2m+1)^2} \sin(2m+1)x
\end{aligned}
$$

Thus we see that

$$T(x,t) = \sum_{m=0}^{\infty} \frac{4(-1)^m}{\pi(2m+1)^2} e^{-\mu(2m+1)^2 t} \sin(2m+1)x$$

satisfies

$$
\begin{aligned}
\frac{\partial T}{\partial t} &= -\mu \frac{\partial^2 T}{\partial x^2} \\
T(0,t) &= T(\pi,t) = 0 \\
T(x,0) &= \theta(x)
\end{aligned}
$$

as desired.

Note that as $t \to \infty, e^{-\mu(2m+1)^2 t} \to 0$. Thus as $t \to \infty, T(x,t) \to 0$. As one might have expected the bar cools towards having a uniform temperature of 0. The same method (develped by Fourier at the start of century 19) allows one in theory to so solve the heat equation with any boundary conditions of this form. In fact we have:

Fact 2.1 *Let $\theta(x)$ be any (reasonable) function on $[0,\pi]$ which vanishes at both end points. Then there is a unique function $T(x,t)$ for $0 \le x \le \pi$, $t \ge 0$ such that satisfies*

$$
\begin{aligned}
\frac{\partial T}{\partial t} &= -\mu \frac{\partial^2 T}{\partial x^2} \\
T(0,t) &= T(\pi,t) = 0 \\
T(x,0) &= \theta(x)
\end{aligned}
$$

Exercises

(1) Solve the equation

$$\frac{\partial T}{\partial t} = \mu \frac{\partial^2 T}{\partial x^2}$$

in $0 \leq x \leq \pi, t \geq 0$, subject to $T(0,t) = T(\pi, t) = 0$ and $T(x,0) = 4 sin\, x$.

(2) Solve the equation

$$\frac{\partial T}{\partial t} = \mu \frac{\partial^2 T}{\partial x^2}$$

in $0 \leq x \leq \pi, t \geq 0$, subject to $T(0,t) = T(\pi, t) = 0$ and

$$T(x,0) = \begin{cases} 0 & x \leq \frac{\pi}{4} \\ 1 & \frac{\pi}{4} < x < \frac{3\pi}{4} \\ 0 & x \geq \frac{3\pi}{4} \end{cases}$$

(You may assume that $T(x,0)$ has a Fourier sine series, which can be computed in the same way as when T is continuous.)

(3) Show that

$$T(x,t) = \frac{100}{\pi} x$$

is a solution to

$$\frac{\partial T}{\partial t} = \mu \frac{\partial^2 T}{\partial x^2}$$

subject to $T(0,t) = 0, T(\pi,t) = 100$.

(4) Solve the equation

$$\frac{\partial T}{\partial t} = \mu \frac{\partial^2 T}{\partial x^2}$$

in $0 \leq x \leq \pi, t \geq 0$, subject to $T(0,t) = 0, T(\pi,t) = 100$,and $T(x,0) = 0$ for $0 \leq x < \pi$. Describe $T(x,t)$ for very large t.
HINT: Look for a solution

$$T(x,t) = \frac{100}{\pi} x + s(x,t)$$

.

(5) Show that if $n = 0, 1, 2, 3, \cdots$ then

$$T(x,t) = e^{-n^2\mu t} \cos nx$$

is a solution of

$$\frac{\partial T}{\partial t} = \mu \frac{\partial^2 T}{\partial x^2}$$

such that

$$\frac{\partial T}{\partial x}(0,t) = \frac{\partial T}{\partial x}(\pi,t) = 0$$

(These boundary condition correspond to a bar which is completely thermally insulated, even at its ends.)

(6) Solve the equation

$$\frac{\partial T}{\partial t} = \mu \frac{\partial^2 T}{\partial x^2}$$

in $0 \le x \le \pi, t \ge 0$, subject to $\cancel{T(0,t) = T(\pi,t) = 0}$, and $T(x,0) = x$. Describe $T(x,t)$ for very large t.

This boundary condition should read: $\frac{\partial T}{\partial x}(0,t) = \frac{\partial T}{\partial x}(\pi,t) = 0$ for $t > 0$.

3 Partial Differential Equation II

We will discuss two other very standard examples of PDE's.

3.1 Laplace's Equation

Consider a square copper plate: $0 \le x \le \pi, 0 \le y \le \pi$. The sides $y = 0, y = \pi$ and $x = 0$ are maintained at a constant temperature of 0. The point (π, y) is maintained at a temperature

$$\begin{array}{rl} y & \text{if} \quad 0 \le y \le \frac{\pi}{2} \\ \pi - y & \text{if} \quad \frac{\pi}{2} \le y \le \pi \end{array}$$

If the plate is in equilibrium, find the temperature distribution on the plate. The temperature $T(x,y,t)$ satisfies

$$\frac{\partial T}{\partial t} = \mu \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right)$$

If the temperature is constant, then we must have

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

This is called *Laplace's equation.*

We must solve Laplace's equation in $0 \le x \le \pi, 0 \le y \le \pi$, subject to $T(x,0) = T(x,\pi) = 0, T(0,y) = 0$,

$$T(\pi, y) = \begin{cases} y & y \le \frac{\pi}{2} \\ \pi - y & y \ge \frac{\pi}{2} \end{cases}$$

Again we look for simple solutions to Laplace's equation of the form

$$T(x,y) = u(x)v(y)$$

We must then solve

$$\begin{array}{ll} u''(x) = au(x) & u(0) = 0 \\ v''(y) = -av(y) & v(0) = v(\pi) = 0 \end{array}$$

As for the heat equation, the only non-trivial solutions are for $a = -n^2$; $n = 1, 2, 3, \cdots$. Then

$$\begin{aligned} v(y) &= A \sin ny \\ u(x) &= B(e^{nx} - e^{-nx}) = 2B \sinh nx \end{aligned}$$

Thus we get the solutions

$$c_n \cdot \sinh nx \sin ny$$

By linearity

$$\sum_{n=1}^{\infty} c_n \sinh nx \sin ny$$

will also be a solution if c_n tend to zero sufficiently fast.

We would like to choose c_n such that

$$\sum_{n=1}^{\infty} c_n \sinh n\pi \, \sin ny = \begin{cases} y & y \le \frac{\pi}{2} \\ \pi - y & y \ge \frac{\pi}{2} \end{cases}$$

As in the last section we see that

$$c_n \sinh n\pi = \begin{cases} 0 & n \text{ even} \\ \frac{4(-1)^{\frac{n-1}{2}}}{\pi n^2} & n \text{ odd} \end{cases}$$

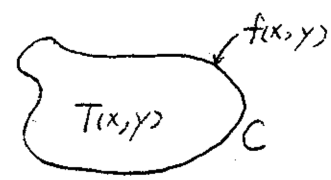
Thus

$$T(x,y) = \sum_{m=0}^{\infty} \frac{4(-1)^m}{\pi(2m+1)^2} \frac{\sinh(2m+1)x}{\sinh(2m+1)\pi} \sin(2m+1)y$$

Fact 3.1 *If C is any smooth simple closed curve in the plane and f is a smooth funciton on C then we can find a function T on the interior of C such that*

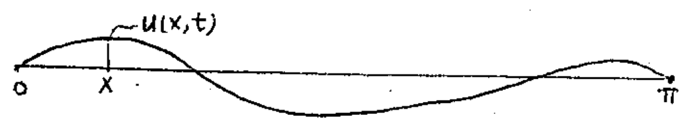
$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

and $T(x, y) = f(x, y)$ for $(x, y) \in C$.



3.2 The Wave Equation

Suppose a violin string of length π is fixed between the points $x = 0$ and $x = \pi$ and suppose the string is plucked with the end points fixed. Describe the movement of the string.



Let $u(x, t)$ denote the displacement of the string from the x-axis at time t and at distance x along the x-axis. Then u satisfies the *wave equation*:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$$

(as long as time is measured in suitable units).

We are looking for solutions satisfying the boundary conditions $u(0, t) = u(\pi, t) = 0$ We look again for simple solutions

$$u(x, t) = v(x)w(t)$$

and obtain the equations

$$\begin{aligned} v''(x) &= av(x) \qquad v(x) = v(\pi) = 0 \\ w''(t) &= av(t) \end{aligned}$$

As in the previoius section, we see we only obtain a non-trivial solution if $a = -n^2$ for n an integer. Then we obtain solutions

$$a_n \sin nt \sin nx \text{ and } b_n \cos nt \sin nx$$

Again linearity gives solutions

$$\sum_{n=0}^{\infty}(a_n \sin nt + b_n \cos nt)\sin nx$$

where we may in fact allow the sums to become infinite if a_n and b_n tend to zero sufficiently fast as $n \to \infty$.

To get a specific solution we must specify what happens at $t = 0$. Suppose that at $t = 0$ the string is stationary with

$$u(x,0) = \begin{cases} \frac{x}{100} & x \leq \frac{\pi}{2} \\ \frac{\pi - x}{100} & x \geq \frac{\pi}{2} \end{cases}$$

Then we require that

$$\sum_{n=1}^{\infty} b_n \sin nx = \begin{cases} \frac{x}{100} & x \leq \frac{\pi}{2} \\ \frac{\pi - x}{100} & x \geq \frac{\pi}{2} \end{cases}$$

As in the last section we see that

$$b_n = \begin{cases} 0 & n \text{ even} \\ \frac{4(-1)^{\frac{n-1}{2}}}{100\pi n^2} & n \text{ odd} \end{cases}$$

What about the a_n? They seem to be arbitrary. The point is the motion of the string depends not only on its initial position, but also on its initial velocity. Using the fact that the string is stationary at $t = 0$ we see that

$$\sum_{n=1}^{\infty}(na_n \cos nt - nb_n \sin nt)\sin nx|_{t=0} = 0$$

Thus

$$\sum_{n=1}^{\infty} na_n \sin nx = 0$$

and so $a_n = 0$ for all n. Thus

$$u(x,t) = \sum_{m=0}^{\infty} \frac{(-1)^m}{25\pi(2m+1)^2} \cos(2m+1)t \, \sin(2m+1)x$$

Notice the difference: The equation

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2}$$

has a unique solution in $0 \le x \le \pi, t \ge 0$ if we specify $T(0,t) = T(\pi,t) = 0$ and we specifty $T(x,0)$. On the other hand, the equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$$

has infinitely many solutions in $0 \le x \le \pi, t \ge 0$ if we specify $u(0,t) = u(\pi,t) = 0$ and we specify $u(x,0)$. In this case we may also specify $\frac{\partial u}{\partial t}(x,0)$.

In general it is a subtle question what boundary conditions we can impose for a PDE and still expect a solution or a unique solution.

Exercises

(1) Suppose that the boundary of a uniform copper disc is maintained at a temperature $T(x,y) = xy$. Find the temperature at the center of the disc when the temperature over the disc is constant in time.

(2) A uniform metal square $0 \le x \le \pi, 0 \le y \le \pi$ has a temperature distribution which is constant in time. If

$$T(0,y) = 0 \qquad T(\pi,y) = \begin{cases} y & y \le \frac{\pi}{2} \\ \pi - y & y \ge \frac{\pi}{2} \end{cases}$$

$$T(x,0) = 0 \qquad T(x,\pi) = \begin{cases} x & x \le \frac{\pi}{2} \\ \pi - x & x \ge \frac{\pi}{2} \end{cases}$$

Find $T(x,y)$ over the whole square.

(3) A violin string fixed at $x = 0$ and $x = \pi$ is initially undisturbed. At time $t = 0$ it is then given a velocity

$$\frac{\partial u}{\partial t}(x,0) = \begin{cases} x & x \le \frac{\pi}{2} \\ \pi - x & x \ge \frac{\pi}{2} \end{cases}$$

Describe the displacement $u(x,t)$ of the string as a function of position and time.

(4) Show that if $f(y)$ and $g(y)$ are any twice differentiable functions then

$$u(x,t) = f(x+t) + g(x-t)$$

satisfies

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$$

If $u(0,t) = u(\pi,t) = 0$, show that we must have

$$\begin{aligned} f(y) &= -g(-y) \\ f(y+2\pi) &= f(y) \\ u(x,t) &= f(x+t) - f(t-x) \end{aligned}$$

If further

$$\frac{\partial u}{\partial t}(x,0) = 0$$

show that $f(y) + f(-y)$ is constant and hence that $f(y) = -f(-y)$. If

$$u(x,0) = \begin{cases} x & x \leq \frac{\pi}{2} \\ \pi - x & x \geq \frac{\pi}{2} \end{cases}$$

find f and hence find $u(x,t)$.

(5) Solve the equation

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

in $0 \leq x \leq \pi, 0 \leq y \leq \pi$, subject to

$$\begin{array}{ll} \frac{\partial T}{\partial x}(0,y) = 0 & \frac{\partial T}{\partial x}(\pi,y) = \sin 2y \\ \frac{\partial T}{\partial y}(x,0) = 0 & \frac{\partial T}{\partial y}(x,\pi) = 0 \end{array}$$

(6) Solve the equation

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

in $0 \leq x \leq \pi, 0 \leq y \leq \pi$, subject to

$$\begin{array}{ll} \frac{\partial T}{\partial x}(0,y) = 0 & \frac{\partial T}{\partial x}(\pi,y) = \sin y \\ \frac{\partial T}{\partial y}(x,0) = 0 & \frac{\partial T}{\partial y}(x,\pi) = 0 \end{array}$$

HINT: Apply Green's theorem to $(\frac{\partial T}{\partial x}, \frac{\partial T}{\partial y})$.