HANDOUT 1: ORDINARY LINEAR DIFFERENTIAL EQUATIONS

Many of the ideas of linear algebra, which we have studied in the context of $\mathbb{R}^n$ or $\mathbb{C}^n$, are applicable much more widely in the mathematical sciences. To try to capture the domain of validity of these methods, mathematicians introduce the concept of "vector space" or "linear space". (These two terms are synonyms.) Rather than studying linear spaces in the abstract, we shall look at some examples which are important in the theory of differential equations. You can read section 9.1 of the textbook if you are interested in the abstract theory. For your convinience we included some references to this section (however, they are not necessary for understanding the content of this handout).

1. Space of smooth functions

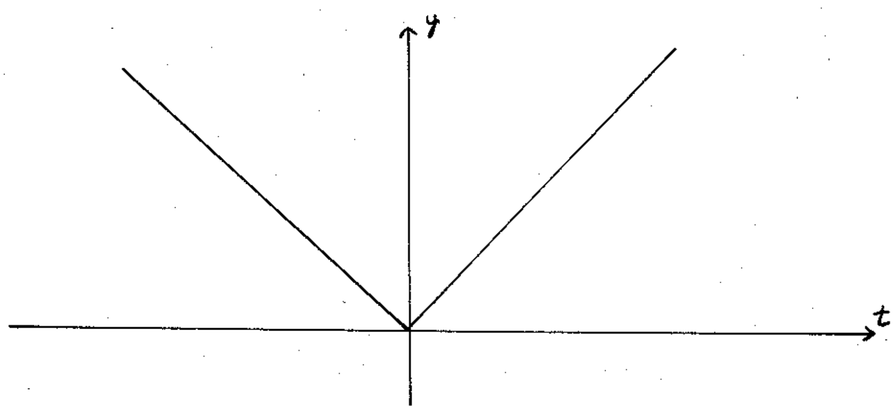
By a *smooth* function from the real numbers to themselves we shall mean a function $f : \mathbb{R} \to \mathbb{R}$ which can be differentiated as many times as you like. We denote by C^∞ the collection of all smooth functions. For instance,

$$f(t) = 1,$$
$$g(t) = t,$$
$$h(t) = e^t$$

are all smooth functions. Indeed,

$$\frac{d^n f}{dt^n} = 0 \text{ for all } n > 0,$$
$$\frac{dg}{dt} = 1, \frac{d^n f}{dt^n} = 0 \text{ for all } n > 1,$$
$$\frac{d^n h}{dt^n} = e^t \text{ for all } n > 0.$$

On the other hand, $f(t) = |t|$ is not a smooth function as it is not even once differentiable at $t = 0$.



If $c \in \mathbb{R}$ and if f and g are smooth functions so is $(cf+g)(t) = cf(t)+g(t)$ (Recall that if f and g are differentiable, so is $cf + g$ and $(cf + g)'(t) = cf'(t) + g'(t)$.) For example, $1 + t$ and $t + 2e^t$ are in our collection C^∞.

Thus, on our collection of functions C^∞ we have defined two operations:

(a) "addition": for two functions f, g in C^∞ we have the function $f+g$ in C^∞ such that for every $t \in \mathbb{R}$ one has

$$(f+g)(t) = f(t) + g(t),$$

(b) "scalar multiplication": for a function f in C^∞ and for a real number c we have the function cf in C^∞ such that

$$(cf)(t) = cf(t).$$

These are the same basic operations that we studied on $\mathbb{R}^n$. Just as most of our study of $\mathbb{R}^n$ was immediately applicable to $\mathbb{C}^n$, so many of the same ideas also apply to our new "space" C^∞. More precisely, C^∞ is a "linear space" in the sense of section 9.1.

Remark. To stress the analogy between $\mathbb{R}^n$ and C^∞ notice that elements of $\mathbb{R}^n$ can be considered as functions $f : \{1, 2, \dots, n\} \to \mathbb{R}$. Indeed, to a vector $x = (x_1, x_2, \dots, x_n)$ we associate the function f such that $f(1) = x_1$, $f(2) = x_2$, ..., $f(n) = x_n$. Thus, $\mathbb{R}^n$ can also be identified with some space of functions. The crucial difference between $\mathbb{R}^n$ and C^∞ is that the former space has dimension n while the latter space is "infinite dimensional" in the sense which will be made more precise later. (It is clear intuitively that an element of C^∞ has infinitely many degrees of freedom.)

Examples. 1. Any polynomial function

$$f(t) = a_n t^n + a_{n-1} t^{n-1} + \ldots + a_1 t + a_0, \tag{1.1}$$

where $a_0, \dots, a_n \in \mathbb{R}$, is smooth and so the collection of all polynomial functions forms a subset P of C^∞. THis subset P has the following two important properties:

(a) if f, g are polynomials so is $f + g$,

(b) if f is a polynomial and c is a real number then cf is a polynomial.

Because P has properties (a) and (b) we call P a *subspace* of C^∞

2. Suppose c_1 and c_2 are real numbers and $c_1 t + c_2 e^t$ is the zero function. Then we claim that $c_1 = c_2 = 0$. Indeed, if $c_2 \neq 0$ then for t very large and positive $c_2 e^t$ will be much larger than $c_1 t$ in magnitude and so $c_1 t + c_2 e^t \neq 0$. Thus, one must have $c_2 = 0$ and hence also $c_1 = 0$. Because of this property we say that t and e^t are *linearly independent.*

3. On the other hand, one can find real numbers c_1, c_2, c_3 such that

$$c_1 \cdot 1 + c_2 \cdot t + c_3 \cdot (1 + t) = 0$$

for all t, namely, $c_1 = 1$, $c_2 = 1$, $c_3 = -1$. Therefore, we say that the functions

$$1, t, \text{ and } (1 + t)$$

are *linearly dependent.*

From these examples we see that the basic concepts of linear algebra make sense for C^∞ instead of $\mathbb{R}^n$. Here are the relevant definitions.

Definition 1.1. A subset $V \subset C^\infty$ is called a *subspace* if for any $f, g \in V$ and every real number c one has $f + g \in V$, $cf \in V$.

Definition 1.2. A *linear combination* of a collection of functions $f_1, f_2, \ldots, f_n \in C^\infty$ is a function of the form $c_1 f_1 + \ldots + c_n f_n$. A collection of functions $(f_1, f_2, ..., f_n)$ is called *linearly dependent* if there exists a non-trivial linear combination $c_1 f_1 + \ldots + c_n f_n$ which is equal to the zero function (where "non-trivial" means that at least one c_i is non-zero).

In C^∞ we sometimes have to work with infinite collections of functions. Here is a sample definition.

Definition 1.3. A *span* of a (not necessary finite) collection of smooth functions $(f_1, f_2, f_3 \ldots)$ is the set of all functions in C^∞ which are linear combinations of a finite number of f_i's. In other words, an element of $span(f_1, f_2, f_3, \ldots)$ has form

$$g(t) = c_1 f_1(t) + c_2 f_2(t) + c_3 f_3(t) + \ldots$$

where only a finite number of c_i's are non-zero.

Examples. 1. The collection $(1, t, t^2, t^3, \ldots)$ spans P but does not span C^∞. Indeed, by definition linear combinations of $(1, t, t^2, t^3, \ldots)$ are polynomial functions (1.1). On the other hand, not all smooth functions are polynomial. For example, e^t doesn't equal to any polynomial function because when $t \to \infty$ the function e^t grows faster than any polynomial function.
2. For any $n > 0$ the functions $(1, t, t^2, \ldots, t^n)$ are linearly independent. Indeed, assume that we have

$$c_0 + c_1 t + c_2 t^2 + \ldots + c_n t^n = 0$$

for all t. If $c_n = 0$ then we have $c_0 + c_1 t + \ldots + c_{n-1} t^{n-1} = 0$ so the problem reduces to the similar one with n replaced by $n - 1$. Therefore, we can assume that $c_n \neq 0$. In this case we can write

$$1 = \frac{c_0 + c_1 t + c_2 t^2 + \ldots c_{n-1} t^{n-1}}{-c_n t^n}$$

which is clearly impossible since the function in the right hand side tends to zero as t goes to infinity.

Since we have the notion of linear independence, we can talk of bases in subspaces of C^∞.

Definition 1.4. Let $V \subset C^\infty$ be a subspace. A (not necessarily finite) collection of functions $(f_1, f_2, \ldots)$ in V is called a *basis* if $V = span(f_1, f_2, \ldots)$ and any finite number of f_i's are linearly independent.

Examples. 1. Let P_n be the set of all polynomial functions of degree at most n. Then P_n is a subspace of C^∞ and $(1, t, \ldots, t^n)$ is a basis of P_n.
2. The infinite collection $(1, t, t^2, t^3, \ldots)$ is a basis of P.

Thus, not every subspace of C^∞ has a finite basis. On the other hand, every subspace $V \subset C^\infty$ which is spanned by a finite number of functions has a basis and the number of elements in all bases is the same. We call such subspaces *finite dimensional* and the number of elements in any basis the dimension of a subspace.

Example. The dimension of P_n is equal to $n+1$, while the subspace P is infinite dimensional.

2. Linear differential operators as linear transformations

The most important techniques we learned in the previous part of the course was the analysis of linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$. In this section we introduce a class of transformations from C^∞ to C^∞ called linear differential operators which are in some respects analogous to these linear transformations.

Let us start with the operation of differentiation. For every function $f \in C^\infty$ we define a new function Df in C^∞ by

$$(Df)(t) = f'(t) = \frac{df}{dt}.$$

For example, $D(\sin t) = \cos t$. The map $D : C^\infty \to C^\infty$ has the following two properties:

(a) $D(f+g) = D(f) + D(g)$ for any $f, g \in C^\infty$,

(b) $D(cf) = cD(f)$ for any $f \in C^\infty$ and a real number c.

Because D has these two properties we call D a *linear transformation*, or simply say that D is linear.

By analogy with the linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$ we can consider the *kernel* of D, that is simply the collection of functions $f(t) \in C^\infty$ such that $D(f) = 0$. But the only functions with zero derivative are the constant functions. Thus, $\ker(D)$ is the collection of all constant functions.

What is the *image* of D? It is the whole of C^∞. Why? If $f(t) \in C^\infty$ then we can define a new function

$$g(t) = \int_0^t f(s)ds.$$

Then $g(t)$ is also a smooth function and by the fundamental theorem of calculus $D(g(t)) = f(t)$.

Remark. If $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation and $\operatorname{Im} T = \mathbb{R}^n$ then $\ker T = 0$. However, $D : C^\infty \to C^\infty$ is a linear transformation and $\operatorname{Im} D = C^\infty$, but $\ker D \neq 0$. This can happen because C^∞ is *infinite dimensional*, i.e. C^∞ can not be spanned by any finite number of elements $f_i(t) \in C^\infty$.

Example. Consider $D^2 = D \circ D : C^\infty \to C^\infty$. In other words,

$$D^2(f)(t) = D(D(f))(t) = f''(t).$$

The kernel of D^2 is the collection of all smooth functions $f(t)$ such that $f''(t) = 0$, i.e. such that $f'(t)$ is a constant function: $f'(t) = a$, i.e. such that

$$f(t) = at + b$$

for some real numbers a and b. Thus, any element $f(t) \in \ker D^2$ is a linear combination of t and 1:

$$f(t) = a \cdot t + b \cdot 1.$$

Being linearly independent t and 1 form a basis in $\ker D^2$. The image of D^2 is the whole of C^∞ (why?).

Suppose that $a_0(t)$, $a_1(t)$, ..., $a_n(t)$, $g(t)$ are smooth functions. Then the equation of the form

$$a_n(t)\frac{d^n f(t)}{dt^n} + \ldots + a_1(t)\frac{df(t)}{dt} + a_0(t)f(t) = g(t) \tag{2.1}$$

is called a *linear ordinary differential equation* for $f(t) \in C^\infty$. If $a_n \neq 0$ then n is called the *order* of this differential equation. To the equation (2.1) we may associate a linear transformation

$$T : C^\infty \to C^\infty$$

defined by

$$T(f)(t) = a_n(t)\frac{d^n f(t)}{dt^n} + \ldots + a_1(t)\frac{df(t)}{dt} + a_0(t)f(t).$$

It is easy to check that T is indeed a linear transformation. The equation (2.1) can be rewritten as

$$T(f) = g. \tag{2.2}$$

If $g \neq 0$ we will call this equation *inhomogeneous.* If $g = 0$ we will call it *homogeneous.* We will refer to the associated equation

$$T(f) = 0$$

as the *associated homogeneous equation.*

If f_0 is any given solution of the equation (2.2) then the general solution is

$$f = f_0 + h$$

where h is a solution of the associated homogeneous equation, i.e. $h \in \ker T$.

In the section 9.4 we will study linear ordinary differential equations *with constant coefficients*, i.e. the equations of the form (2.1) with functions $a_i(t)$ which are constant. The corresponding homogeneous equations are closely related to continuous dynamical systems we considered in section 8. Indeed, consider the equation of the form

$$\frac{d^n f(t)}{dt^n} + a_{n-1}\frac{d^{n-1} f}{dt^{n-1}} + \ldots + a_1\frac{df(t)}{dt} + a_0 f(t) = 0 \tag{2.3}$$

where $a_0, a_1, \dots, a_{n-1}$ are given constants. Then we can introduce a vector-valued function $x(t) = (x_0(t), x_1(t), \dots, x_{n-1}(t))$ by setting

$$\begin{aligned} x_0(t) &= f(t), \\ x_1(t) &= \frac{df(t)}{dt}, \\ &\dots \\ x_{n-1}(t) &= \frac{d^{n-1}f(t)}{dt}. \end{aligned}$$

We can recover $f(t)$ from $x(t)$ since $f(t) = x_0(t)$. On the other hand, the equation (2.3) is equivalent to the following system of linear differential equations *of the first order.*

$$\begin{aligned} \frac{dx_0(t)}{dt} &= x_1(t), \\ \frac{dx_1(t)}{dt} &= x_2(t), \\ &\dots \\ \frac{dx_{n-1}(t)}{dt} &= -a_{n-1}x_{n-1}(t) - \dots - a_1x_1(t) - a_0x_0(t). \end{aligned}$$

We can write this system in the matrix form:

$$\frac{dx(t)}{dt} = Ax(t) \tag{2.4}$$

where A is the following matrix:

$$A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \dots & 1 \\ -a_0 & -a_1 & -a_2 & \dots & -a_{n-1} \end{pmatrix}$$

Thus, the solutions to the equation (2.3) are in one-to-one correspondence with the solutions of the linear dynamical system (2.4). In particular, a solution to (2.4) is uniquely determined by its initial state $x(0)$. This means that a function $f(t)$ satisfying the differential equation (2.3) of order n is uniquely determined by the values of $f(0)$, $\frac{df}{dt}(0)$, ..., $\frac{d^{n-1}f}{dt}(0)$.

Recall that our study of the system (2.4) was based on finding the eigenvalues of matrix A. Thus, it is important to calculate the characteristic polynomial of A. It is easy to see that it is equal to

$$f_A(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \dots + a_1\lambda + a_0.$$

For this reason this polynomial is called the *characteristic polynomial* of the differential equation (2.3), or of the corresponding differential operator $D^n + a_{n-1}D^{n-1} + \dots + a_1D + a_0$.