Mathematics 21b

First Exam Solutions

Your Section (circle one):

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Question	Points	Score
1	20	
2	8	
3	7	
4	7	
5	8	
Total		

The exam will last 90 minutes.

No calculators are allowed.

Justify your answers carefully (except in Question 1).

For Questions 2–5, no credit can be given for unsubstantiated answers.

Write your final answers in the spaces provided.

(1) True or False (no explanation is necessary).

(T) **F** : There is a linear transformation of the plane which takes the triangle on the left to the triangle on the right.
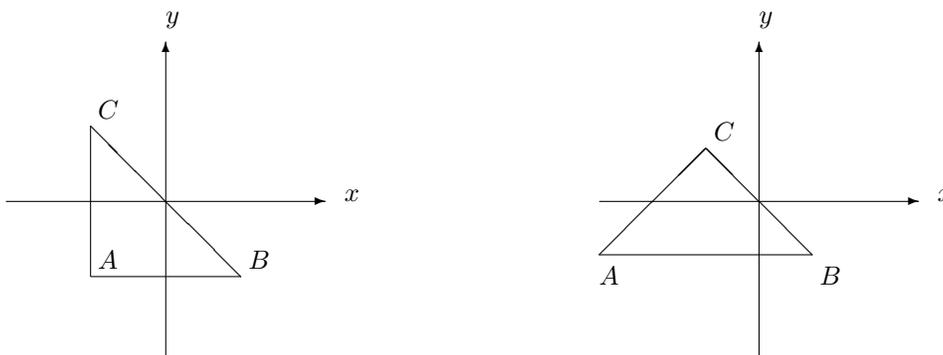
To see this, note that the vectors A and B are linearly independent, so there is a unique linear transformation taking A and B on the left to A and B on the right. Since $C = -B$, we have that $S(C) = -S(B)$, as desired.

(T) **F** : Let A and B denote the 3×3 matrices representing respectively reflection in the plane $z = x + y$ and orthogonal projection onto the plane $z = x + y$. Then $AB = BA$.

Since, from a geometric standpoint, it does not matter in which order these two operations are performed, we can apply A and B in either order and obtain the same matrix (hench linear transformation) as a result.

T (F) : There is a linear transformation with exactly two distinct points in its image.

Recall for this and the following question that both the image and the kernel of a linear transformation are subspaces, and that two distinct points do not form a subspace.

T (F) : There is a linear transformation with exactly two distinct points in its kernel.

(T) **F** : If $T : \mathbf{R}^3 \to \mathbf{R}^3$ is a linear transformation then the set of vectors $\vec{x} \in \mathbf{R}^3$ with $T\vec{x} = \vec{x}$ is a subspace.

For $T\vec{0} = \vec{0}$, and if $T\vec{x} = \vec{x}$ and $T\vec{y} = \vec{y}$, then $T(\vec{x} + \vec{y}) = T\vec{x} + T\vec{y} = \vec{x} + \vec{y}$, and if $c \in \mathbf{R}$ then $T(c\vec{x}) = cT\vec{x} = c\vec{x}$, so all three conditions for a subspace are satisfied.

T Ⓕ **:** The vectors

$$\begin{bmatrix} -1 \\ 2 \\ 3 \\ -4 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ -3 \end{bmatrix}, \begin{bmatrix} -5 \\ 1 \\ 3 \\ 1 \end{bmatrix},$$

span the subspace $x + y + z + w = 0$ of $\mathbf{R}^4$.

Note that this subspace is three-dimensional and that each of these vectors satisfies the equation $x + y + z + w = 0$, but that these vectors are linearly dependent (in fact $\begin{bmatrix} -5 \\ 1 \\ 3 \\ 1 \end{bmatrix} = 2\begin{bmatrix} -1 \\ 2 \\ 3 \\ -4 \end{bmatrix} - 3\begin{bmatrix} 1 \\ 1 \\ 1 \\ -3 \end{bmatrix}$), so they span only two dimensions of the solution space and hence do not form a basis.

Ⓣ **F :** If $\{\vec{v}_1, ..., \vec{v}_n\}$ is a basis of $\mathbf{R}^n$ and if $T : \mathbf{R}^n \to \mathbf{R}^n$ is an invertible linear transformation then $\{T\vec{v}_1, ..., T\vec{v}_n\}$ is also a basis of $\mathbf{R}^n$.

Consider the matrix of T with respect to the basis $\vec{v}_1, \ldots, \vec{v}_n$. Since T is invertible the columns of the matrix representation of T must be linearly independent, and since there are n of them the vectors $T\vec{v}_1, \ldots, T\vec{v}_n$ form a basis for $\mathbf{R}^n$.

T Ⓕ **:** There is a linear transformation $T : \mathbf{R}^3 \to \mathbf{R}^3$ with $\operatorname{Im} T = \ker T$.

For, $\dim(\operatorname{Im}T) + \dim(\ker T) = 3$.

T Ⓕ **:** There are two matrices A and B such that A has rank 3, B has rank 100 and AB is defined and has rank 4.

Since B is applied first, the rank of AB is *at most* the rank of A, despite the fact that B has very large rank.

Ⓣ **F :** There are two matrices A and B such that A has rank 3, B has rank 100 and AB is defined and has rank 2.

By the same reasoning, it is possible that AB has rank two, as 98 of the basis vectors of the image of B may lie in the kernel of A.

(2) Let A be a 5×5 matrix. Suppose a finite number of elementary row operations reduce A to the following matrix.

$$\begin{bmatrix} 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 & 1 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

(a) Find a basis of the kernel of A.

Solving $A\vec{x} = \vec{0}$ gives that $\ker A$ is given by $\begin{bmatrix} t \\ 0 \\ 0 \\ t \\ 0 \end{bmatrix}$, so $\ker A$ is spanned

by (for instance) the single vector $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$. Since a single vector is automatically linearly independent, this vector gives a basis of $\ker A$.

(b) Suppose the elementary row operations used to reduce A to the form given above were, in the order they were applied, the following:

    i) Add row 2 to row 3.

    ii) Swap row 2 and row 4.

    iii) Multiply row 4 by 1/2.

    iv) Subtract row 1 from row 5.

Find a basis of the image of A.

We know that the image of A is spanned by the columns of A that have leading 1's in the reduced row-echelon form of A. We see from the partially reduced form of A given in the previous part that the first, third, and fourth columns will contain leading 1's in the reduced row-echelon form of A.

We now have to find A itself. Reversing each of the row operations from the last applied to the first gives

$$
\begin{bmatrix}
0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
-1 & 0 & 0 & 1 & 1 \\
0 & -1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0
\end{bmatrix} \widetilde{\text{vi}^{-1}}
\begin{bmatrix}
0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
-1 & 0 & 0 & 1 & 1 \\
0 & -1 & 0 & 0 & 0 \\
0 & -1 & 1 & 0 & 0
\end{bmatrix} \widetilde{\text{iii}^{-1}}
$$

$$
\begin{bmatrix}
0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
-1 & 0 & 0 & 1 & 1 \\
0 & -2 & 0 & 0 & 0 \\
0 & -1 & 1 & 0 & 0
\end{bmatrix} \widetilde{\text{ii}^{-1}}
\begin{bmatrix}
0 & -1 & 0 & 0 & 0 \\
0 & -2 & 0 & 0 & 0 \\
-1 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 \\
0 & -1 & 1 & 0 & 0
\end{bmatrix} \widetilde{\text{i}^{-1}}
$$

$$
\begin{bmatrix}
0 & -1 & 0 & 0 & 0 \\
0 & -2 & 0 & 0 & 0 \\
-1 & 2 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 \\
0 & -1 & 1 & 0 & 0
\end{bmatrix} = A
$$

Thus $\operatorname{Im} A$ is spanned by $\begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$. Since the third vector is a scalar multiple of the first, we only need the first two vectors to form a basis of $\operatorname{Im} A$.

(3) (a) Find a basis for the plane $x + 2y + z = 0$ in $\mathbf{R}^3$.

Since we have a plane, any two linearly independent vectors that lie in the plane will form a basis for it. By inspection the vectors $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ fulfill these conditions, so they form a basis for the plane.

(Note: There are many possible solutions to this part.)

(b) Find a 3×3 matrix which represents (with respect to the standard basis of $\mathbf{R}^3$) a linear transformation with image the plane $x+2y+z = 0$ and with kernel the line $x = y = z$.

One such matrix is $\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 1 & -2 \end{bmatrix}$. Notice that the columns of this

matrix span the plane, and that $\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

(4) Consider the three vectors

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}.$$

You may assume that they are each perpendicular to the other two and each have length 3.

Also consider

$$S = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}.$$

You may assume that S is invertible and that $S^{-1} = 1/9\ S$.

Let $T : \mathbf{R}^3 \to \mathbf{R}^3$ denote rotation through $90°$ about $\vec{v}_1$, clockwise as you face $\vec{0}$ from $\vec{v}_1$.

(a) Find the matrix of T with respect to the basis $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ of $\mathbf{R}^3$.

Note first that since $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$ are orthogonal, rotation about $\vec{v}_1$ takes place in the $\vec{v}_2$-$\vec{v}_3$ plane. Inspection of the graph of the three vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ gives that rotation through $90°$ about $\vec{v}_1$ will take $\vec{v}_1 \to \vec{v}_1$, $\vec{v}_2 \to -\vec{v}_3$, and $\vec{v}_3 \to \vec{v}_2$. So the matrix of T with respect to the basis $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is

$$[T]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}.$$

(b) Find the matrix of T with respect to the standard basis of $\mathbf{R}^3$.

This we find just by multiplying $[T]_{\mathcal{B}}$ on the left and on the right by the appropriate change of basis matrices. We see that the matrix S given in the problem is the change of basis matrix from $\mathcal{B}$ to the standard basis E. So the matrix of T with respect to the standard basis is

$$[T]_E = S[T]_{\mathcal{B}}S^{-1}$$

$$= S[T]_{\mathcal{B}}\frac{1}{9}S$$

$$= \frac{1}{9}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} 1 & 8 & -4 \\ -4 & 4 & 7 \\ 8 & 1 & 4 \end{bmatrix}.$$

(5) Let S denote the unit square with vertices at

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

(a) Find the matrix C (with respect to the standard basis) of the shear parallel to the y-axis which takes $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ to $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and sketch the image of S under C.

If the shear is parallel to the y-axis, then the shear takes $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ to $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

The matrix of S with respect to the standard basis is then

$$C = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

The image of S under C is sketched below.
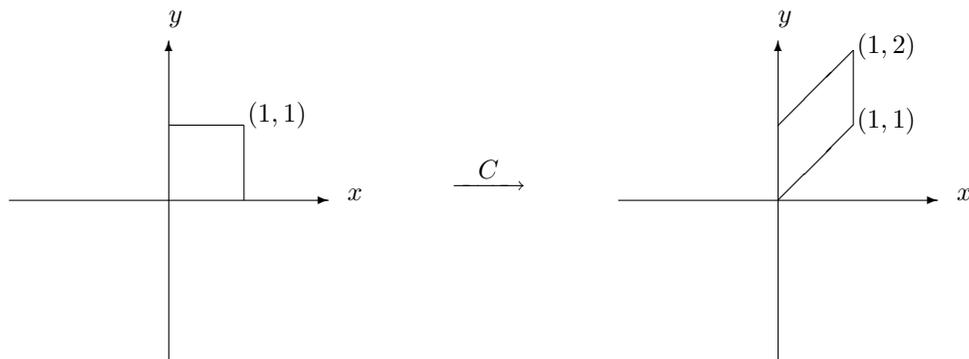
(b) Find 2×2 matrices A and B each representing a shear such that ABC represents a counterclockwise rotation through $90°$. Sketch the image of S under BC and ABC.

Let B be the matrix of the shear parallel to the line $x = y$ that takes $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ to $\begin{bmatrix} -1 \\ 0 \end{bmatrix}$, and let A be the matrix of the shear parallel to the x-axis taking $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ to $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$. The matrices A and B are then

$$A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}.$$

and the images of S under BC and ABC are: